Нормираност на водородоподобните вълнови функции φ_{2p_x}, φ_{2p_y}, и φ_{2p_z}

2p$_x$-AO: $\varphi_{2p_x} = \frac{1}{4\sqrt{2\pi}}.r.e^{-r/2}\sin\theta\cos\varphi$

$$\int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{\infty}\left(\frac{1}{4\sqrt{2\pi}}.r.e^{-r/2}\sin\theta\cos\varphi\right)^2 r^2\sin\theta dr d\theta d\varphi =$$

$$= \frac{1}{32\pi}\int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{\infty} r^4 e^{-r} dr \sin^3\theta\cos^2\varphi d\theta d\varphi =$$

Табличен интеграл:

$$\int_0^{\infty} x^n e^{-ax} dx = \frac{n!}{a^{n+1}}$$

тогава:

$$\int_0^{\infty} r^4 e^{-r} dr = \frac{4!}{1^{4+1}} = 24$$

$$= \frac{24}{32\pi}\int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi} \sin^3\theta d\theta\cos^2\varphi d\varphi =$$

$$= \frac{24}{32.12\pi}\int_{\varphi=0}^{2\pi}(\cos 3\theta - 9\cos\theta)\Big|_0^{\pi}\cos^2\varphi d\varphi =$$

Интегралът:

$$\int \sin^3 x dx = \frac{1}{12}(\cos 3x - 9\cos x) + i$$

съгласно:
http://integrals.wolfram.com/index.jsp

$$= \frac{1}{16\pi}\int_{\varphi=0}^{2\pi}[(\cos 3\pi - 9\cos\pi) - (\cos 3.0 - 9\cos 0)]\cos^2\varphi d\varphi = = \frac{1}{16\pi}\int_{\varphi=0}^{2\pi}[-1+9-1+9]\cos^2\varphi d\varphi =$$

$$= \frac{16}{16\pi}\int_{\varphi=0}^{2\pi}\cos^2\varphi d\varphi = \frac{1}{\pi}\int_{\varphi=0}^{2\pi}\cos^2\varphi d\varphi =$$

$$= \frac{1}{2\pi}(\varphi + \sin\varphi\cos\varphi)\Big|_0^{2\pi} = \frac{1}{2\pi}[2\pi + \sin 2\pi\cos 2\pi - 0 - \sin 0\cos 0] =$$

$$= \frac{1}{2\pi}[2\pi + 0 - 0 - 0] = \mathbf{1}$$

Интегралът:

$$\int \cos^2 x dx = \frac{1}{2}(x + \sin x\cos x) + i$$

съгласно:
http://integrals.wolfram.com/index.jsp

2p$_y$-AO: $\varphi_{2p_x} = \frac{1}{4\sqrt{2\pi}}.r.e^{-r/2}\sin\theta\sin\varphi$

$$\int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{\infty}\left(\frac{1}{4\sqrt{2\pi}}.r.e^{-r/2}\sin\theta\sin\varphi\right)^2 r^2\sin\theta dr d\theta d\varphi =$$

$$= \frac{1}{32\pi}\int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{\infty} r^4 e^{-r} dr \sin^3\theta\sin^2\varphi d\theta d\varphi =$$

Табличен интеграл:

$$\int_0^{\infty} x^n e^{-ax} dx = \frac{n!}{a^{n+1}}$$

тогава:

$$\int_0^{\infty} r^4 e^{-r} dr = \frac{4!}{1^{4+1}} = 24$$

$$= \frac{24}{32\pi}\int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi}\sin^3\theta d\theta\sin^2\varphi d\varphi =$$

$$= \frac{24}{32.12\pi}\int_{\varphi=0}^{2\pi}(\cos 3\theta - 9\cos\theta)\Big|_0^{\pi}\sin^2\varphi d\varphi =$$

Интегралът:

$$\int \sin^3 x dx = \frac{1}{12}(\cos 3x - 9\cos x) + i$$

съгласно:
http://integrals.wolfram.com/index.jsp

$$= \frac{1}{16\pi}\int_{\varphi=0}^{2\pi}[(\cos 3\pi - 9\cos\pi) - (\cos 3.0 - 9\cos 0)]\sin^2\varphi d\varphi = = \frac{1}{16\pi}\int_{\varphi=0}^{2\pi}[-1+9-1+9]\sin^2\varphi d\varphi =$$

$$= \frac{16}{16\pi}\int_{\varphi=0}^{2\pi}\sin^2\varphi d\varphi = \frac{1}{\pi}\int_{\varphi=0}^{2\pi}\sin^2\varphi d\varphi =$$

$$= \frac{1}{2\pi}(\varphi - \sin\varphi\cos\varphi)\Big|_0^{2\pi} = \frac{1}{2\pi}[2\pi - \sin 2\pi\cos 2\pi - 0 + \sin 0\cos 0] =$$

$$= \frac{1}{2\pi}[2\pi - 0 - 0 + 0] = \mathbf{1}$$

Интегралът:

$$\int \sin^2 x dx = \frac{1}{2}(x - \sin x\cos x) + i$$

съгласно:
http://integrals.wolfram.com/index.jsp

2p$_z$-AO: $\varphi_{2p_x} = \frac{1}{4\sqrt{2\pi}}.r.e^{-r/2}\cos\theta$

$$\int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{\infty}\left(\frac{1}{4\sqrt{2\pi}}.r.e^{-r/2}\cos\theta\right)^2 r^2\sin\theta dr d\theta d\varphi =$$

$$= \frac{1}{32\pi}\int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{\infty} r^4 e^{-r} dr\cos^2\theta\sin\theta d\theta d\varphi =$$

$$= \frac{24}{32\pi}\int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi}\cos^2\theta\sin\theta d\theta d\varphi =$$

$$= -\frac{24}{32\pi}\int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi}\cos^2\theta d(\cos\theta) d\varphi =$$

$$= -\frac{24}{3.32\pi}\int_{\varphi=0}^{2\pi}\cos^3\theta\Big|_0^{\pi} d\varphi = -\frac{24}{96\pi}\int_{\varphi=0}^{2\pi}(\cos^3\pi - \cos^3 0)d\varphi =$$

$$= -\frac{1}{4\pi}\int_{\varphi=0}^{2\pi}(-1-1)d\varphi = \frac{1}{2\pi}\int_{\varphi=0}^{2\pi} d\varphi = \frac{1}{2\pi}\varphi\Big|_0^{2\pi} =$$

$$= \frac{1}{2\pi}[2\pi - 0] = \mathbf{1}$$

Табличен интеграл:

$$\int_0^{\infty} x^n e^{-ax} dx = \frac{n!}{a^{n+1}}$$

тогава:

$$\int_0^{\infty} r^4 e^{-r} dr = \frac{4!}{1^{4+1}} = 24$$

Табличен интеграл:

$$\int x^2 dx = \frac{1}{3}x^3 + i$$

тогава:

$$\int_0^{\pi} \cos^2\theta d(\cos\theta) = \frac{1}{3}\cos^3\theta\Big|_0^{\pi}$$