Преобразуване на тригонометрични изрази. Тригонометрични уравнения и неравенства

1) Докажете тъждествата:

$$\boxed{\begin{aligned}\sin 2\alpha &= 2\sin\alpha\cos\alpha\\ \cos 2\alpha &= \cos^2\alpha - \sin^2\alpha\end{aligned}}$$

а) $$\frac{\sin\alpha + \cos\alpha}{\cos\alpha - \sin\alpha} = \operatorname{tg} 2\alpha + \frac{1}{\cos 2\alpha}$$

Имаме $$\operatorname{tg} 2\alpha + \frac{1}{\cos 2\alpha} = \frac{\sin 2\alpha}{\cos 2\alpha} + \frac{1}{\cos 2\alpha} = \frac{2\sin\alpha\cos\alpha + \sin^2\alpha + \cos^2\alpha}{\cos^2\alpha - \sin^2\alpha}$$

$$= \frac{(\cos\alpha + \sin\alpha)^{\cancel{2}}}{(\cos\alpha - \sin\alpha)\cancel{(\cos\alpha + \sin\alpha)}} = \frac{\sin\alpha + \cos\alpha}{\cos\alpha - \sin\alpha}.$$

б) $$\frac{1 + \sin\alpha - 2\sin^2\left(45^\circ - \frac{\alpha}{2}\right)}{4\cos\frac{\alpha}{2}} = \sin\frac{\alpha}{2}$$

Съгласно формулата $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$ преобразуваме $\sin\left(45^\circ - \frac{\alpha}{2}\right) = \frac{\sqrt{2}}{2}\cos\frac{\alpha}{2} - \frac{\sqrt{2}}{2}\sin\frac{\alpha}{2}$ и лявата страна приема вида

$$\frac{1 + \sin\alpha - 2\sin^2\left(45^\circ - \frac{\alpha}{2}\right)}{4\cos\frac{\alpha}{2}} = \frac{1 + \sin\alpha - 2\cdot\frac{2}{4}\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)^2}{4\cos\frac{\alpha}{2}}$$

$$= \frac{\cancel{1} + 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} - \cancel{1} + 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{4\cos\frac{\alpha}{2}} = \sin\frac{\alpha}{2}.$$

в) $$\frac{\sin 4\alpha}{1 + \cos 4\alpha} \cdot \frac{\cos 2\alpha}{1 + \cos 2\alpha} \cdot \frac{\cos\alpha}{1 + \cos\alpha} = \operatorname{tg}\frac{\alpha}{2}$$

Използваме, че $1 + \cos 2\alpha = 2\cos^2\alpha \Rightarrow$

$$\frac{\sin 4\alpha}{1 + \cos 4\alpha} \cdot \frac{\cos 2\alpha}{1 + \cos 2\alpha} \cdot \frac{\cos\alpha}{1 + \cos\alpha} = \frac{\cancel{2}\sin 2\alpha\cancel{\cos 2\alpha}}{\cancel{2\cos^2 2\alpha}} \cdot \frac{\cancel{\cos 2\alpha}}{2\cos^2\alpha} \cdot \frac{\cos\alpha}{2\cos^2\frac{\alpha}{2}} =$$

$$= \frac{\cancel{2}\sin\alpha\cancel{\cos\alpha}\cdot\cancel{\cos\alpha}}{\cancel{2\cos^2\alpha}\cdot 2\cos^2\frac{\alpha}{2}} = \frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2\cos^2\frac{\alpha}{2}} = \operatorname{tg}\frac{\alpha}{2}.$$

2) Пресметнете стойността на израза $\dfrac{\cos\alpha+\sin\alpha}{\sin\alpha-\cos\alpha}$, ако $\sin\alpha\cos\alpha=\dfrac{2}{5}$.

От $\sin\alpha\cos\alpha=\dfrac{2}{5} \Rightarrow 2\sin\alpha\cos\alpha=\dfrac{4}{5} \Rightarrow$

- $1+2\sin\alpha\cos\alpha=\sin^2\alpha+\cos^2\alpha+2\sin\alpha\cos\alpha=(\sin\alpha+\cos\alpha)^2=\dfrac{9}{5} \Rightarrow$
  $\sin\alpha+\cos\alpha=\pm\dfrac{3}{\sqrt5}$.
- $1-2\sin\alpha\cos\alpha=\sin^2\alpha+\cos^2\alpha-2\sin\alpha\cos\alpha=(\sin\alpha-\cos\alpha)^2=\dfrac{1}{5} \Rightarrow$
  $\sin\alpha-\cos\alpha=\pm\dfrac{1}{\sqrt5}$.

$\Rightarrow \dfrac{\sin\alpha+\cos\alpha}{\sin\alpha-\cos\alpha}=\pm 3.$

3) Намерете най-малката и най-голямата стойност на функцията:

а) $y=3\cos^2 3x+5\sin^2 3x=\underbrace{3\cos^2 3x+3\sin^2 3x}_{=1}+2\sin^2 3x \Rightarrow$

$\Rightarrow y=2\sin^2 3x+1$

От $0\le\sin^2 3x\le 1 \Rightarrow 1\le y\le 3.$

б) $y=3\cos x+4\sin x \Rightarrow y=5\left(\dfrac{3}{5}\cos x+\dfrac{4}{5}\sin x\right)$

От $\left(\dfrac{3}{5}\right)^2+\left(\dfrac{4}{5}\right)^2=1$, $\dfrac{3}{5},\dfrac{4}{5}\in[-1;1] \Rightarrow$ съществува ъгъл α, за който $\sin\alpha=\dfrac{3}{5}$, $\cos\alpha=\dfrac{4}{5}$

$\Rightarrow y=5(\sin\alpha\cos x+\cos\alpha\sin x)=5\sin(x+\alpha) \Rightarrow$

от $-1\le\sin(x+\alpha)\le 1 \Rightarrow -5\le y=5\sin(x+\alpha)\le 5.$

в) $y=\dfrac{\sin^2 2x}{\sin^4 x+\cos^4 x}$

От $\sin^2 x+\cos^2 x=1 \;\big|^2 \Leftrightarrow (\sin^2 x+\cos^2 x)^2=1 \Leftrightarrow$
$\sin^4 x+2\cos^2 x\sin^2 x+\cos^4 x=1 \Leftrightarrow \sin^4 x+\cos^4 x=1-2\cos^2 x\sin^2 x$

$\Rightarrow \sin^4 x+\cos^4 x=1-\dfrac{1}{2}\sin^2 2x \Rightarrow$

$y=\dfrac{\sin^2 2x}{1-\frac{1}{2}\sin^2 2x}=\dfrac{2\sin^2 2x}{2-\sin^2 2x}$

Полагаме $\sin^2 2x = t \Rightarrow \boxed{0 \le t \le 1} \Rightarrow$

$$y(t) = \frac{2t}{2-t} \Rightarrow y' = \frac{2(2-t)+2t}{(2-t)^2} = \frac{4}{(2-t)^2} > 0$$

$\Rightarrow y$ е строго растяща функция $\Rightarrow$

$y_{\text{нмс}} = y(0) = 0$ и $y_{\text{нгс}} = y(1) = 2$.

4) Докажете, че за $\forall x \in \mathbb{R}$ е в сила

$$\frac{1}{2} \le \sin^4 x + \cos^4 x \le 1.$$

От установеното в 3) в) $\sin^4 x + \cos^4 x = 1 - \frac{1}{2}\sin^2 2x$ и $0 \le \sin^2 2x \le 1 \Rightarrow$ най-малката стойност на израза $\sin^4 x + \cos^4 x$ се приема при най-голямата стойност на $\sin^2 2x$, т.е. при $\sin^2 2x = 1$ и $\Rightarrow$ $\sin^4 x + \cos^4 x \ge 1 - \frac{1}{2}\cdot 1 = \frac{1}{2}$, а най-голямата стойност на същия израз се приема при най-малката стойност на $\sin^2 2x$, т.е. при $\sin^2 2x = 0 \Rightarrow \sin^4 x + \cos^4 x \le 1$.

*5) Докажете, че $\underline{-\sqrt{2} \le \sin x + \cos x \le \sqrt{2}}$.

$$(\sin x + \cos x)^2 = \sin^2 x + \cos^2 x + 2\sin x\cos x = 1 + \sin 2x \Rightarrow$$

$\underline{0 \le (\sin x + \cos x)^2 \le 2}$, тъй като $\sin 2x \le 1 \Rightarrow$

след коренуване получаваме $-\sqrt{2} \le \sin x + \cos x \le \sqrt{2}$.

6) $$\cos x - \sin x = \sin\left(\frac{\pi}{2} - x\right) - \sin x = 2\sin\frac{\frac{\pi}{2} - x - x}{2}\cos\frac{\frac{\pi}{2} - x + x}{2} =$$
$$= 2\sin\left(\frac{\pi}{4} - x\right)\cdot\frac{\sqrt{2}}{2} = \sqrt{2}\sin\left(\frac{\pi}{4} - x\right).$$

* аналогично важи и за $\sin x - \cos x$ (както и $\cos x - \sin x$).
Можем да запишем

$$-\sqrt{2} \le \sin x \pm \cos x \le \sqrt{2},$$

което ще имаме предвид в задачи, в които ще използваме полагането $t = \sin x \pm \cos x$.

Основни тригонометрични уравнения

1) $\sin x + \sin 2x + \sin 3x = \cos x + \cos 2x + \cos 3x$

$$2\sin\frac{x+3x}{2}\cos\frac{x-3x}{2} + \sin 2x = 2\cos\frac{x+3x}{2}\cos\frac{x-3x}{2} + \cos 2x$$

$$2\sin 2x\cos x + \sin 2x = 2\cos 2x\cos x + \cos 2x$$

$$\sin 2x(2\cos x + 1) = \cos 2x(2\cos x + 1) \Leftrightarrow$$

$$(2\cos x + 1)(\sin 2x - \cos 2x) = 0$$

$2\cos x + 1 = 0 \quad \cup \quad \sin 2x = \cos 2x$

$\Updownarrow \qquad\qquad \Updownarrow$

$\cos x = -\frac{1}{2} \Leftrightarrow \qquad \operatorname{tg} 2x = 1$

$\Updownarrow$

$\boxed{x = \pm\frac{2\pi}{3} + 2k\pi} \qquad 2x = \frac{\pi}{4} + l\pi \Leftrightarrow \boxed{x = \frac{\pi}{8} + l\frac{\pi}{2}}$

$k = 0, \pm 1, \pm 2, \ldots \qquad l = 0, \pm 1, \pm 2, \ldots$

2) $\sin(x+1)\cos(2x+2) = \sin(3x+3)\cos(4x+4)$

$$\frac{1}{2}\left[\sin(3x+3) + \cancel{\sin(-x-1)}\right] = \frac{1}{2}\left[\sin(7x+7) + \cancel{\sin(-x-1)}\right]$$

$$\sin(3x+3) - \sin(7x+7) = 0 \Leftrightarrow$$

$$\sin(7x+7) - \sin(3x+3) = 0$$

$$2\sin\frac{4x+4}{2}\cos\frac{10x+10}{2} = 0 \Leftrightarrow \sin(2x+2)\cos(5x+5) = 0$$

$\Leftrightarrow \sin(2x+2) = 0 \quad \cup \quad \cos(5x+5) = 0$

$\Updownarrow \qquad\qquad \Updownarrow$

$2x + 2 = k\pi \qquad 5x + 5 = \frac{\pi}{2} + l\pi$

$\Updownarrow$

$x = k\frac{\pi}{2} - 1 \qquad x = \frac{\pi}{10} + l\frac{\pi}{5} - 1$

$k = 0, \pm 1, \pm 2, \ldots \qquad l = 0, \pm 1, \pm 2, \ldots$

3) $4\sin x \sin\left(x+\frac{\pi}{3}\right) = 1$

$$2\left[\cos\left(x - x - \frac{\pi}{3}\right) - \cos\left(x + x + \frac{\pi}{3}\right)\right] = 1$$

$$2\left[\cos\frac{\pi}{3} - \cos\left(2x+\frac{\pi}{3}\right)\right] = 1 \Leftrightarrow 2\left[\frac{1}{2} - \cos\left(2x+\frac{\pi}{3}\right)\right] = 1$$

$$1 - 2\cos\left(2x+\frac{\pi}{3}\right) = 1 \Leftrightarrow \cos\left(2x+\frac{\pi}{3}\right) = 0 \Leftrightarrow$$

$$2x + \frac{\pi}{3} = \frac{\pi}{2} + k\pi \Leftrightarrow 2x = \frac{\pi}{6} + k\pi \Leftrightarrow x = \frac{\pi}{12} + k\frac{\pi}{2},$$

$k = 0, \pm 1, \pm 2, \dots$

4) $\sin 7x + \sin 3x = 3\cos 2x$

$$2\sin 5x \cos 2x - 3\cos 2x = 0$$

$$\cos 2x\,(2\sin 5x - 3) = 0$$

$\cos 2x = 0 \quad \cup \quad 2\sin 5x = 3 \Leftrightarrow \sin 5x = \frac{3}{2} > 1$ ↯

н. р. и.

$\Updownarrow$

$$2x = \frac{\pi}{2} + k\pi \Leftrightarrow x = \frac{\pi}{4} + k\frac{\pi}{2}$$

$k = 0, \pm 1, \pm 2, \dots$

5) $\sin 2x + \sin 3x + \sin 4x + \sin 5x = 0$

$$2\sin 3x \cos x + 2\sin 4x \cos x = 0$$

$$\cos x\,(\sin 3x + \sin 4x) = 0$$

$$\cos x \cdot \sin\frac{7x}{2} \cdot \cos\frac{x}{2} = 0$$

$\cos x = 0 \quad \cup \quad \sin\frac{7x}{2} = 0 \quad \cup \quad \cos\frac{x}{2} = 0 \Leftrightarrow$

$\Updownarrow$ $\qquad$ $\Updownarrow$

$$x = \frac{\pi}{2} + k\pi \qquad \frac{7x}{2} = l\pi \qquad \frac{x}{2} = \frac{\pi}{2} + n\pi$$

$$x = \frac{2l\pi}{7} \qquad x = \pi + 2n\pi = (2n+1)\pi$$

6) $\sin 4x \sin 2x + \cos 2x \cos 4x = \cos 2x \sin 2x$

$\cos 2x - \cos 2x \sin 2x = 0$

$\cos 2x (\sin 2x - 1) = 0$

$\swarrow \qquad \searrow$

$\cos 2x = 0 \qquad\qquad \sin 2x = 1$

$\Updownarrow \qquad\qquad \Updownarrow$

$2x = \frac{\pi}{2} + k\pi \qquad\qquad 2x = \frac{\pi}{2} + 2k\pi$

$x = \frac{\pi}{4} + k\frac{\pi}{2} \leftarrow \qquad x = \frac{\pi}{4} + k\pi$

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7) $\sin 2x + \operatorname{tg} x = 2 \iff \sin 2x + \operatorname{tg} x = 1 + 1$

$\operatorname{tg} x - 1 = 1 - \sin 2x \iff \frac{\sin x}{\cos x} - 1 = \sin^2 x + \cos^2 x - 2\sin x \cos x$

$\iff \frac{\sin x - \cos x}{\cos x} = (\sin x - \cos x)^2$

$\swarrow \qquad \searrow$

$\sin x - \cos x = 0$

$\Updownarrow$

$\sin x = \cos x$

$\Updownarrow$

$\operatorname{tg} x = 1 \iff \underline{x = \frac{\pi}{4} + k\pi} \in$ Д.О.

Д.О. $\cos x \neq 0 \iff$

$x \neq \frac{\pi}{2} + k\pi$

$\frac{1}{\cos x} = \sin x - \cos x \quad \Big| \cdot \cos x \neq 0$

$1 = \sin x \cos x - \cos^2 x$

$1 + \cos^2 x = \frac{1}{2} \cdot 2\sin x \cos x$

$1 + \cos^2 x = \frac{1}{2} \sin 2x$

л. страна $= 1 + \cos^2 x \geq 1$

д. страна $= \frac{1}{2} \sin 2x \leq \frac{1}{2}$ $\Rightarrow$

няма решения

8) $\frac{1}{\cos x} - \operatorname{tg} x = \sin x + \cos x$, Д.О. $\cos x \neq 0$

$1 - \sin x = \sin x \cos x + \cos^2 x$

$\cancel{\cos^2 x} + \sin^2 x - \sin x = \sin x \cos x \cancel{+ \cos^2 x}$

$\sin x (\sin x - 1 - \cos x) = 0$

$\swarrow \qquad \searrow$

$\sin x = 0 \qquad\qquad \sin x - \cos x = 1.$

$x = k\pi \qquad\qquad x_1 = \frac{\pi}{2} + 2l\pi, \quad x_2 = (2l+1)\pi$