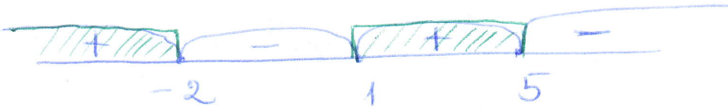
Рационални неравенства – метод на интервалите

Основни примери – цели рационални неравенства

а) $(x+2)(x-1)(x-5) > 0$

$x \in (-2; 1) \cup (5; +\infty)$

б) $(x+2)(x-1)\underline{(5-x)} \geq 0$

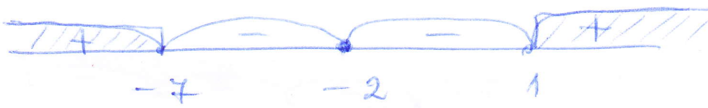


$x \in (-\infty; -2] \cup [1; 5]$

в) $(x+7)(x+2)^2(x-1)^3 \leq 0$

$x \in [-7; 1]$

г) $(x+7)\underline{(x+2)^2}(x-1)^3 \geq 0$

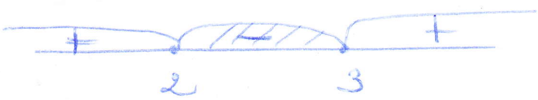


$x \in (-\infty; -7] \cup \{-2\} \cup [1; +\infty)$

д) $(x-2)(x-3)(x^2-x+3) < 0$

Тъй като дискриминантата D на кв. тричлен $f(x) = x^2 - x + 3$ е отрицателна, то $f(x) > 0$ за $\forall x \in \mathbb{R} \Rightarrow (x-2)(x-3)(x^2-x+3) < 0 \Leftrightarrow$

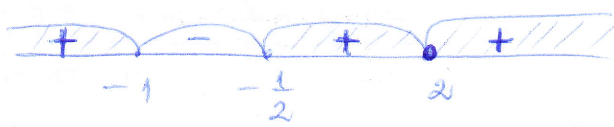
$(x-2)(x-3) < 0$



$x \in (2; 3)$

положителен старши коефициент $a = 1$

е) $(x+1)^3(x-2)^2\left(x+\frac{1}{2}\right)^5>0$



$x\in(-\infty;-1)\cup\left(-\frac{1}{2};2\right)\cup(2;+\infty)$

ж) $(x-1)(x^2-1)(x^3-1)(x^4-1)\le 0$

$\underline{(x-1)}\,\underline{(x-1)}(x+1)\underline{(x-1)}(x^2+x+1)(x^2+1)\underline{(x-1)}(x+1)\le 0$

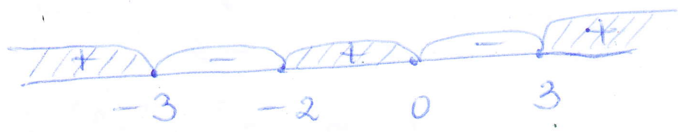
$(x-1)^4(x+1)^2\underbrace{(x^2+x+1)(x^2+1)}_{\text{квадратни тричлени с отрицателни дискриминанти}}\le 0 \iff$

$(x-1)^4(x+1)^2\le 0 \iff \underline{x=\pm 1}$

з) $\left|\begin{array}{l}(x^3-9x)(x+2)>0\\(x+1)(x-3)<0\end{array}\right. \iff \left|\begin{array}{l}x(x^2-9)(x+2)>0\\(x+1)(x-3)<0\end{array}\right.$

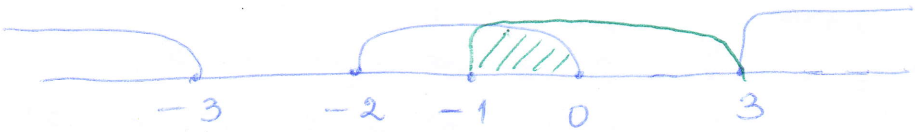
$\iff \left|\begin{array}{ll}x(x-3)(x+3)(x+2)>0 & (1)\\(x+1)(x-3)<0 & (2)\end{array}\right.$

(1)



(2)

сечението на (1) и (2) е



$x\in(-1;0)$

г) $(x+3)^3 + (x-5)^3 \ge 2(x-1)^3$

$\underbrace{(x+3)^3 - (x-1)^3} + \underbrace{(x-5)^3 - (x-1)^3} \ge 0$

$(x+3-x+1)\left[(x+3)^2 + (x+3)(x-1) + (x-1)^2\right] +$

$+ (x-5-x+1)\left[(x-5)^2 + (x-5)(x-1) + (x-1)^2\right] \ge 0$

$4\left(x^2 + 6x + 9 + x^2 + \cancel{2x} - 3 + x^2 - \cancel{2x} + 1\right)$

$-4\left(x^2 - 10x + 25 + x^2 - 6x + 5 + x^2 - 2x + 1\right) \ge 0 \;\Big|\, :4$

$\cancel{3x^2} + 6x + 7 - \cancel{3x^2} + 18x - 31 \ge 0$

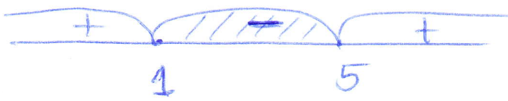
$24x - 24 \ge 0 \quad \Leftrightarrow \quad \boxed{x \ge 1}$

й) $(3x^2 - 2x)^2 + 12x + 5 < 18x^2$

$(3x^2 - 2x)^2 - 6(3x^2 - 2x) + 5 < 0$

Полагаме $3x^2 - 2x = t \;\Rightarrow\; t^2 - 6t + 5 < 0$

$\Leftrightarrow (t-5)(t-1) < 0$



$\Rightarrow \underline{1 < t < 5} \;\Rightarrow$

$1 < 3x^2 - 2x < 5 \;\Leftrightarrow\; \left|\begin{array}{l} 3x^2 - 2x - 1 > 0 \\ 3x^2 - 2x - 5 < 0 \end{array}\right. \Leftrightarrow$

$\Leftrightarrow \left|\begin{array}{ll} 3(x-1)\left(x+\frac{1}{3}\right) > 0 & (1) \\ 3\left(x-\frac{5}{3}\right)(x+1) < 0 & (2) \end{array}\right.$

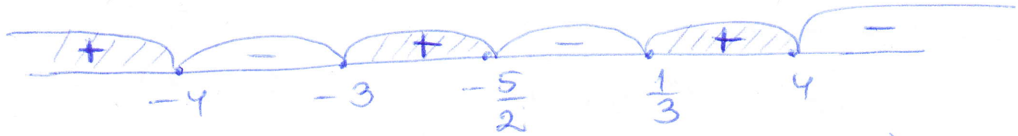
$\left.\begin{array}{l} (1) \Leftrightarrow x \in \left(-\infty; -\frac{1}{3}\right) \cup (1; +\infty) \\ (2) \Leftrightarrow x \in \left(-1; \frac{5}{3}\right) \end{array}\right\}$ сечение

$\Rightarrow$

$x \in \left(-1; -\frac{1}{3}\right) \cup \left(1; \frac{5}{3}\right)$

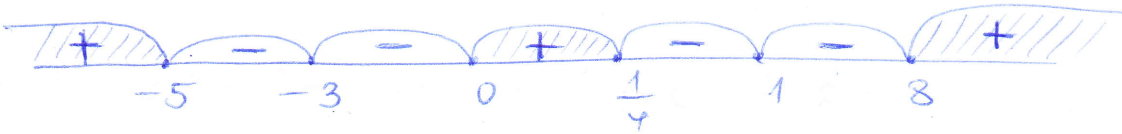
Дробно-рационални неравенства – метод на интервалите

а) $\dfrac{(x+3)(4-x)(2x+5)}{(3x-1)(x+4)} > 0$



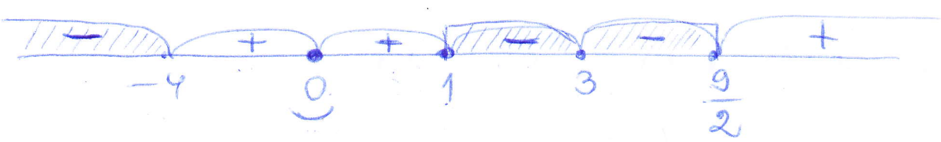
$x \in (-\infty; -4) \cup \left(-3; -\frac{5}{2}\right) \cup \left(\frac{1}{3}; 4\right)$

б) $\dfrac{x^3(x-1)^4(x+5)}{(4x-1)(x+3)^2(x-8)} > 0$



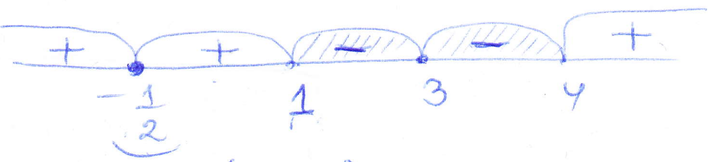
$x \in (-\infty; -5) \cup \left(0; \frac{1}{4}\right) \cup (8; +\infty)$

в) $\dfrac{x^2(2x-9)(x-1)^3}{(x+4)^5(2x-6)^4} \le 0$



$x \in (-\infty; -4) \cup \{0\} \cup [1; 3) \cup \left(3; \frac{9}{2}\right]$

г) $\dfrac{(2x+1)^6(x-3)^4}{(x-4)(x-1)^7} \le 0$



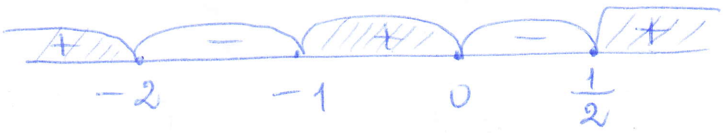
$x \in \left\{-\frac{1}{2}\right\} \cup (1; 4)$

g) $2+\dfrac{3}{x+1} > \dfrac{2}{x} \Leftrightarrow 2+\dfrac{3}{x+1}-\dfrac{2}{x} > 0$

$$\frac{2x(x+1)+3x-2(x+1)}{x(x+1)} > 0$$

$$\frac{2x^2+\cancel{2x}+3x-\cancel{2x}-2}{x(x+1)} > 0 \Leftrightarrow \frac{2x^2+3x-2}{x(x+1)} > 0$$

$$\frac{2(x+2)\left(x-\frac{1}{2}\right)}{x(x+1)} > 0$$



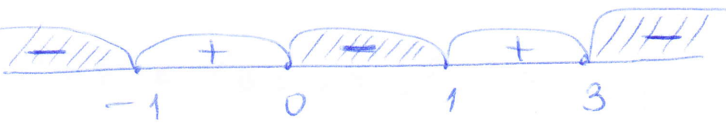
$x \in (-\infty; -2) \cup (-1; 0) \cup \left(\frac{1}{2}; +\infty\right)$

e) $\dfrac{1}{x-1}+\dfrac{2}{x+1}-\dfrac{3}{x} < 0$

$$\frac{x(x+1)+2x(x-1)-3(x-1)(x+1)}{x(x-1)(x+1)} < 0$$

$$\frac{\cancel{x^2}+x+\cancel{2x^2}-2x-\cancel{3x^2}+3}{x(x-1)(x+1)} < 0$$

$$\frac{3-x}{x(x-1)(x+1)} < 0$$



$x \in (-\infty; -1) \cup (0; 1) \cup (3; +\infty)$

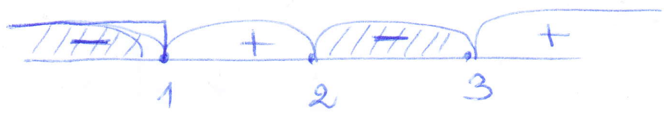
14) $$\frac{x^2-2x-1}{x-2}+\frac{2}{x-3}\le x$$

$$\underbrace{\frac{x^2-2x-1}{x-2}-x}+\frac{2}{x-3}\le 0$$

$$\frac{\cancel{x^2}-\cancel{2x}-1-\cancel{x^2}+\cancel{2x}}{x-2}+\frac{2}{x-3}\le 0$$

$$\frac{2}{x-3}-\frac{1}{x-2}\le 0 \iff \frac{2(x-2)-(x-3)}{(x-2)(x-3)}\le 0$$

$$\frac{x-1}{(x-2)(x-3)}\le 0$$



$x\in(-\infty;1]\cup(2;3)$

3) $$x-1+\frac{2x-2}{x-3}-\frac{7}{x^2-2x-3}>2$$

$$x-3+\frac{2x-2}{x-3}-\frac{7}{(x+1)(x-3)}>0$$

$$\frac{(x-3)^2(x+1)+2(x-1)(x+1)-7}{(x-3)(x+1)}>0$$

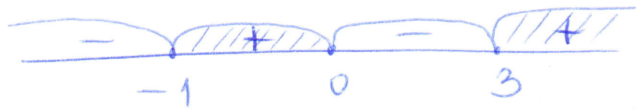
$$\frac{(x+1)(x^2-6x+9)+2x^2-9}{(x-3)(x+1)}>0$$

$$\frac{x^3-6x^2+9x+x^2-6x+\cancel{9}+2x^2-\cancel{9}}{(x-3)(x+1)}>0$$

$$\frac{x^3-3x^2+3x}{(x-3)(x+1)}>0 \iff \frac{x(x^2-3x+3)}{(x-3)(x+1)}>0$$

Дискриминантата на кв. тричлен x^2-3x+3 е
$D = 9-12 = -3 < 0 \Rightarrow x^2-3x+3 > 0$ за $\forall x \in \mathbb{R}$.
$\Rightarrow$ неравенството е еквивалентно на

$$\frac{x}{(x-3)(x+1)} > 0$$



$x \in (-1;0) \cup (3;+\infty)$

Намерете стойностите на реалния параметър a, за които неравенството

$$\frac{x^2+ax-1}{2x^2-2x+3} < 1$$

е изпълнено за $\forall x \in \mathbb{R}$.

Тъй като $2x^2-2x+3 > 0$ за $\forall x \in \mathbb{R}$, то даденото неравенство е еквивалентно на

$$x^2+ax-1 < 2x^2-2x+3 \iff$$

$$x^2-(a+2)x+4 > 0.$$

Тъй като старшият коефициент е положителен, то горното неравенство е изпълнено за $\forall x \in \mathbb{R}$ $\iff D < 0$.

$$D = (a+2)^2-16 = (a+6)(a-2) < 0 \iff$$

$$a \in (-6;2)$$