Lie groups as four-dimensional special complex manifolds with Norden metric

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Abstract

An example of a four-dimensional special complex manifold with Norden metric of constant holomorphic sectional curvature is constructed via a two-parametric family of solvable Lie algebras. The curvature properties of the obtained manifold are studied. Necessary and sufficient conditions for the manifold to be isotropic Kählerian are given.
2000 Mathematics Subject Classification: 53C15, 53C50.
Keywords: almost complex manifold, Norden metric, Lie group, Lie algebra.

1 Preliminaries

Let (M, J, g) be a $2n$-dimensional almost complex manifold with Norden metric, i.e. J is an almost complex structure and g is a metric on M such that:

$$J^2x = -x, \qquad g(Jx, Jy) = -g(x, y), \qquad x, y \in \mathfrak{X}(M). \tag{1.1}$$

The associated metric $\widetilde{g}$ of g on M, given by $\widetilde{g}(x, y) = g(x, Jy)$, is a Norden metric, too. Both metrics are necessarily neutral, i.e. of signature (n, n).

If ∇ is the Levi-Civita connection of g, the tensor field F of type $(0, 3)$ is defined by

$$F(x, y, z) = g\left((\nabla_x J)y, z\right) \tag{1.2}$$

and has the following symmetries

$$F(x, y, z) = F(x, z, y) = F(x, Jy, Jz). \tag{1.3}$$

Let $\{e_i\}$ $(i = 1, 2, \ldots, 2n)$ be an arbitrary basis of T_pM at a point p of M. The components of the inverse matrix of g are denoted by g^{ij} with respect to the basis $\{e_i\}$. The Lie 1-forms θ and θ^* associated with F are defined by, respectively

$$\theta(x) = g^{ij}F(e_i, e_j, x), \qquad \theta^* = \theta \circ J. \tag{1.4}$$

The Nijenhuis tensor field N for J is given by

$$N(x, y) = [Jx, Jy] - [x, y] - J[Jx, y] - J[x, Jy]. \tag{1.5}$$

It is known [4] that the almost complex structure is complex iff it is integrable, i.e. $N = 0$.

A classification of the almost complex manifolds with Norden metric is introduced in [2], where eight classes of these manifolds are characterized according to the properties of F. The three basic classes: $\mathcal{W}_1$, $\mathcal{W}_2$ of *the special complex manifolds with Norden metric* and $\mathcal{W}_3$ of *the quasi-Kähler manifolds with Norden metric* are given as follows:

$$
(1.6)\qquad
\begin{aligned}
&\mathcal{W}_1 : F(x,y,z) = \tfrac{1}{2n}\left[g(x,y)\theta(z) + g(x,z)\theta(y)\right.\\
&\qquad\qquad\qquad \left.+g(x,Jy)\theta(Jz) + g(x,Jz)\theta(Jy)\right];\\
&\mathcal{W}_2 : F(x,y,Jz) + F(y,z,Jx) + F(z,x,Jy) = 0, \quad \theta = 0 \;\Leftrightarrow\; N = 0, \quad \theta = 0;\\
&\mathcal{W}_3 : F(x,y,z) + F(y,z,x) + F(z,x,y) = 0.
\end{aligned}
$$

The class $\mathcal{W}_0$ of *the Kähler manifolds with Norden metric* is defined by $F = 0$ and is contained in each of the other classes.

Let R be the curvature tensor of ∇, i.e. $R(x,y)z = \nabla_x\nabla_y z - \nabla_y\nabla_x z - \nabla_{[x,y]}z$. The corresponding (0,4)-type tensor is defined by $R(x,y,z,u) = g\left(R(x,y)z,u\right)$. The Ricci tensor ρ and the scalar curvatures τ and τ^* are given by:

$$
(1.7)\qquad \rho(y,z) = g^{ij}R(e_i,y,z,e_j), \qquad \tau = g^{ij}\rho(e_i,e_j), \qquad \tau^* = g^{ij}\rho(e_i,Je_j).
$$

A tensor of type (0,4) is said to be *curvature-like* if it has the properties of R. Let S be a symmetric (0,2)-tensor. We consider the following curvature-like tensors:

$$
(1.8)\qquad
\begin{aligned}
&\psi_1(S)(x,y,z,u) = g(y,z)S(x,u) - g(x,z)S(y,u)\\
&\qquad\qquad\qquad\quad + g(x,u)S(y,z) - g(y,u)S(x,z),\\
&\pi_1 = \tfrac{1}{2}\psi_1(g), \quad \pi_2(x,y,z,u) = g(y,Jz)g(x,Ju) - g(x,Jz)g(y,Ju).
\end{aligned}
$$

It is known that on a pseudo-Riemannian manifold M ($\dim M = 2n \geq 4$) the conformal invariant Weyl tensor has the form

$$
(1.9)\qquad W(R) = R - \frac{1}{2(n-1)}\big\{\psi_1(\rho) - \frac{\tau}{2n-1}\pi_1\big\}.
$$

Let $\alpha = \{x,y\}$ be a non-degenerate 2-plane spanned by the vectors $x,y \in T_pM$, $p \in M$. The sectional curvature of α is given by

$$
(1.10)\qquad k(\alpha;p) = \frac{R(x,y,y,x)}{\pi_1(x,y,y,x)}.
$$

We consider the following basic sectional curvatures in T_pM with respect to the structures J and g: *holomorphic sectional curvatures* if $J\alpha = \alpha$ and *totally real sectional curvatures* if $J\alpha \perp \alpha$ with respect to g.

The square norm of ∇J is defined by $\|\nabla J\|^2 = g^{ij}g^{kl}g\left((\nabla_{e_i}J)e_k,(\nabla_{e_j}J)e_l\right)$. Then, by (1.2) we get

$$
(1.11)\qquad \|\nabla J\|^2 = g^{ij}g^{kl}g^{pq}F_{ikp}F_{jlq},
$$

where $F_{ikp} = F(e_i,e_k,e_p)$.

An almost complex manifold with Norden metric satisfying the condition $\|\nabla J\|^2 = 0$ is called an *isotropic Kähler manifold with Norden metric* [3].

2 Almost complex manifolds with Norden metric of constant holomorphic sectional curvature

In this section we obtain a relation between the vanishing of the holomorphic sectional curvature and the vanishing of $\|\nabla J\|^2$ on $\mathcal{W}_2$-manifolds and $\mathcal{W}_3$-manifolds with Norden metric.

In [1] it is proved the following

Theorem A. ([1]) *An almost complex manifold with Norden metric is of pointwise constant holomorphic sectional curvature if and only if*

$$\begin{aligned} &3\{R(x,y,z,u)+R(x,y,Jz,Ju)+R(Jx,Jy,z,u)+R(Jx,Jy,Jz,Ju)\}\\ &-R(Jy,Jz,x,u)+R(Jx,Jz,y,u)-R(y,z,Jx,Ju)+R(x,z,Jy,Ju)\\ &-R(Jx,z,y,Ju)+R(Jy,z,x,Ju)-R(x,Jz,Jy,u)+R(y,Jz,Jx,u)\\ &=8H\{\pi_1+\pi_2\} \end{aligned} \tag{2.1}$$

for some $H \in FM$ and all $x,y,z,u \in \mathfrak{X}(M)$. In this case $H(p)$ is the holomorphic sectional curvature of all holomorphic non-degenerate 2-planes in T_pM, $p \in M$. ■

Taking into account (1.7) and (1.8), the total trace of (2.1) implies

$$H(p) = \frac{1}{4n^2}(\tau + \tau^{**}), \tag{2.2}$$

where $\tau^{**} = g^{il}g^{jk}R(e_i, e_j, Je_k, Je_l)$.

In [5] we have proved that on a $\mathcal{W}_2$-manifold it is valid

$$\|\nabla J\|^2 = 2(\tau + \tau^{**}), \tag{2.3}$$

and in [3] it is proved that on a $\mathcal{W}_3$-manifold

$$\|\nabla J\|^2 = -2(\tau + \tau^{**}). \tag{2.4}$$

Then, by Theorem A, (2.2), (2.3) and (2.4) we obtain

Theorem 2.1. *Let (M,J,g) be an almost complex manifold with Norden metric of pointwise constant holomorphic sectional curvature $H(p)$, $p \in M$. Then*

(i) $\|\nabla J\|^2 = 8n^2H(p)$ *if* $(M,J,g) \in \mathcal{W}_2$;

(ii) $\|\nabla J\|^2 = -8n^2H(p)$ *if* $(M,J,g) \in \mathcal{W}_3$. ■

Theorem 2.1 implies

Corollary 2.2. *Let (M,J,g) be a $\mathcal{W}_2$-manifold or $\mathcal{W}_3$-manifold of pointwise constant holomorphic sectional curvature $H(p)$, $p \in M$. Then, (M,J,g) is isotropic Kählerian iff $H(p) = 0$.*

In the next section we construct an example of a $\mathcal{W}_2$-manifold of constant holomorphic sectional curvature.

3 Lie groups as four-dimensional $\mathcal{W}_2$-manifolds

Let $\mathfrak{g}$ be a real 4-dimensional Lie algebra corresponding to a real connected Lie group G. If $\{X_1, X_2, X_3, X_4\}$ is a basis of left invariant vector fields on G and $[X_i, X_j] = C_{ij}^k X_k$ $(i, j, k = 1, 2, 3, 4)$ then the structural constants C_{ij}^k satisfy the anti-commutativity condition $C_{ij}^k = -C_{ji}^k$ and the Jacobi identity $C_{ij}^k C_{ks}^l + C_{js}^k C_{ki}^l + C_{si}^k C_{kj}^l = 0$.

We define an almost complex structure J and a compatible metric g on G by the conditions, respectively:

$$JX_1 = X_3, \quad JX_2 = X_4, \quad JX_3 = -X_1, \quad JX_4 = -X_2, \tag{3.1}$$

$$\begin{aligned} &g(X_1, X_1) = g(X_2, X_2) = -g(X_3, X_3) = -g(X_4, X_4) = 1,\\ &g(X_i, X_j) = 0, \quad i \neq j, \quad i, j = 1, 2, 3, 4. \end{aligned} \tag{3.2}$$

Because of (1.1), (3.1) and (3.2) g is a Norden metric. Thus, (G, J, g) is a 4-dimensional almost complex manifold with Norden metric.

From (3.2) it follows that the well-known Levi-Civita identity for g takes the form

$$2g(\nabla_{X_i} X_j, X_k) = g([X_i, X_j], X_k) + g([X_k, X_i], X_j) + g([X_k, X_j], X_i). \tag{3.3}$$

Let us denote $F_{ijk} = F(X_i, X_j, X_k)$. Then, by (1.2) and (3.3) we have

$$\begin{aligned} 2F_{ijk} = g\big([X_i, JX_j] - J[X_i, X_j], X_k\big) + g\big(J[X_k, X_i] - [JX_k, X_i], X_j\big)\\ +g\big([X_k, JX_j] - [JX_k, X_j], X_i\big). \end{aligned} \tag{3.4}$$

According to (1.6) to construct an example of a $\mathcal{W}_2$-manifold we need to find sufficient conditions for the Nijenhuis tensor N and the Lie 1-form θ to vanish on $\mathfrak{g}$.

By (1.2), (1.5), (3.2) and (3.4) we compute the essential components N_{ij}^k $(N(X_i, X_j) = N_{ij}^k X_k)$ of N and $\theta_i = \theta(X_i)$ of θ, respectively, as follows:

$$\begin{aligned} N_{12}^1 &= C_{34}^1 - C_{12}^1 - C_{23}^3 + C_{14}^3, &\quad \theta_1 &= 2C_{13}^1 - C_{12}^4 + C_{14}^2 + C_{23}^2 - C_{34}^4,\\ N_{12}^2 &= C_{34}^2 - C_{12}^2 - C_{23}^4 + C_{14}^4, &\quad \theta_2 &= 2C_{24}^2 + C_{12}^3 + C_{14}^1 + C_{23}^1 + C_{34}^3,\\ N_{12}^3 &= C_{34}^3 - C_{12}^3 + C_{23}^1 - C_{14}^1, &\quad \theta_3 &= 2C_{13}^3 + C_{12}^2 + C_{14}^4 + C_{23}^4 + C_{34}^2,\\ N_{12}^4 &= C_{34}^4 - C_{12}^4 + C_{23}^2 - C_{14}^2, &\quad \theta_4 &= 2C_{24}^4 - C_{12}^1 + C_{14}^3 + C_{23}^3 - C_{34}^1. \end{aligned} \tag{3.5}$$

Then, (1.6) and (3.5) imply

Theorem 3.1. *Let (G, J, g) be a 4-dimensional almost complex manifold with Norden metric defined by (3.1) and (3.2). Then, (G, J, g) is a $\mathcal{W}_2$-manifold iff for the Lie algebra $\mathfrak{g}$ of G are valid the conditions*:

$$\begin{aligned} C_{13}^1 &= C_{12}^4 - C_{23}^2 = C_{34}^4 - C_{14}^2, &\quad C_{13}^3 &= -\left(C_{12}^2 + C_{23}^4\right) = -\left(C_{14}^4 + C_{34}^2\right),\\ C_{24}^4 &= C_{12}^1 - C_{14}^3 = C_{34}^1 - C_{23}^3, &\quad C_{24}^2 &= -\left(C_{12}^3 + C_{14}^1\right) = -\left(C_{23}^1 + C_{34}^3\right), \end{aligned} \tag{3.6}$$

where C_{ij}^k $(i, j, k = 1, 2, 3, 4)$ satisfy the Jacodi identity. ■

One solution to (3.6) and the Jacobi identity is the 2-parametric family of solvable Lie algebras $\mathfrak{g}$ given by

$$\mathfrak{g}: \begin{array}{ll} [X_1,X_2]=\lambda X_1-\lambda X_2, & [X_2,X_3]=\mu X_1+\lambda X_4, \\ [X_1,X_3]=\mu X_2+\lambda X_4, & [X_2,X_4]=\mu X_1+\lambda X_3, \\ [X_1,X_4]=\mu X_2+\lambda X_3, & [X_3,X_4]=-\mu X_3+\mu X_4, \qquad \lambda,\mu\in\mathbb{R}. \end{array} \tag{3.7}$$

Let us study the curvature properties of the $\mathcal{W}_2$-manifold (G,J,g), where the Lie algebra $\mathfrak{g}$ of G is defined by (3.7).

By (3.2), (3.3) and (3.7) we obtain the components of the Levi-Civita connection:

$$\begin{array}{ll} \nabla_{X_1}X_2=\lambda X_1+\mu(X_3+X_4), & \nabla_{X_2}X_1=\lambda X_2+\mu(X_3+X_4), \\ \nabla_{X_3}X_4=-\lambda(X_1+X_2)-\mu X_3, & \nabla_{X_4}X_3=-\lambda(X_1+X_2)-\mu X_4, \\ \nabla_{X_1}X_1=-\lambda X_2, \quad \nabla_{X_2}X_2=-\lambda X_1, & \nabla_{X_3}X_3=\mu X_4, \quad \nabla_{X_4}X_4=\mu X_3, \\ \nabla_{X_1}X_3=\nabla_{X_1}X_4=\mu X_2, & \nabla_{X_2}X_3=\nabla_{X_2}X_4=\mu X_1, \\ \nabla_{X_3}X_1=\nabla_{X_3}X_2=-\lambda X_4, & \nabla_{X_4}X_1=\nabla_{X_4}X_2=-\lambda X_3. \end{array} \tag{3.8}$$

Taking into account (3.4) and (3.7) we compute the essential non-zero components of F:

$$\begin{array}{l} F_{114}=-F_{214}=F_{312}=\frac{1}{2}F_{322}=\frac{1}{2}F_{411}=F_{412}=-\lambda, \\ F_{112}=\frac{1}{2}F_{122}=\frac{1}{2}F_{211}=F_{212}=-F_{314}=F_{414}=\mu. \end{array} \tag{3.9}$$

The other non-zero components of F are obtained from (1.3).

By (1.11) and (3.9) for the square norm of ∇J we get

$$\|\nabla J\|^2=-32(\lambda^2-\mu^2). \tag{3.10}$$

Further, we obtain the essential non-zero components $R_{ijks}=R(X_i,X_j,X_k,X_s)$ of the curvature tensor R as follows:

$$\begin{array}{l} -\frac{1}{2}R_{1221}=-R_{1341}=-R_{2342}=R_{3123}=\frac{1}{2}R_{3443}=R_{4124}=\lambda^2+\mu^2, \\ R_{1331}=R_{1441}=R_{2332}=R_{2442}=-R_{1324}=-R_{1423}=\lambda^2-\mu^2, \\ R_{1231}=R_{1241}=R_{2132}=R_{2142} \\ =-R_{3143}=-R_{3243}=-R_{4134}=-R_{4234}=2\lambda\mu. \end{array} \tag{3.11}$$

Then, by (1.7) and (3.11) we get the components $\rho_{ij}=\rho(X_i,X_j)$ of the Ricci tensor and the values of the scalar curvatures τ and τ^*:

$$\begin{array}{ll} \rho_{11}=\rho_{22}=-4\lambda^2, & \rho_{33}=\rho_{44}=-4\mu^2, \\ \rho_{12}=\rho_{34}=-2(\lambda^2+\mu^2), & \rho_{13}=\rho_{14}=\rho_{23}=\rho_{24}=4\lambda\mu, \\ \tau=-8(\lambda^2-\mu^2), & \tau^*=16\lambda\mu. \end{array} \tag{3.12}$$

Let us consider the characteristic 2-planes α_{ij} spanned by the basic vectors $\{X_i,X_j\}$: totally real 2-planes - α_{12}, α_{14}, α_{23}, α_{34} and holomorphic 2-planes - α_{13}, α_{24}. By (1.10) and (3.11) for the sectional curvatures of the holomorphic 2-planes we obtain

$$k(\alpha_{13})=k(\alpha_{24})=-(\lambda^2-\mu^2). \tag{3.13}$$

Then it is valid

Theorem 3.2. *The manifold (G, J, g) is of constant holomorphic sectional curvature.*

Using (1.9), (3.11) and (3.12) for the essential non-zero components $W_{ijks} = W(X_i, X_j, X_k, X_s)$ of the Weyl tensor W we get:

$$\begin{aligned} &\tfrac{1}{2}W_{1221} = W_{1331} = W_{1441} = W_{2332} = W_{2442} = \tfrac{1}{2}W_{3443} \\ &= -\tfrac{1}{3}W_{1324} = -\tfrac{1}{3}W_{1423} = \tfrac{1}{3}(\lambda^2 - \mu^2). \end{aligned} \tag{3.14}$$

Finally, by (1.9), (3.10), (3.12), (3.13) and (3.14) we establish the truthfulness of

Theorem 3.3. *The following conditions are equivalent*:

(i) (G, J, g) *is isotropic Kählerian*;

(ii) $|\lambda| = |\mu|$;

(iii) $\tau = 0$;

(iv) (G, J, g) *is of zero holomorphic sectional curvature*;

(v) *the Weyl tensor vanishes.*

(vi) $R = \frac{1}{2}\psi_1(\rho)$.

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