ON THE CURVATURE PROPERTIES OF REAL TIME-LIKE HYPERSURFACES OF KÄHLER MANIFOLDS WITH NORDEN METRIC

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A type of almost contact hypersurfaces with Norden metric of a Kähler manifold with Norden metric is considered. The curvature tensor and the special sectional curvatures are characterized. The canonical connection on such manifolds is studied and the form of the corresponding Kähler curvature tensor is obtained. Some curvature properties of the manifolds belonging to the widest integrable main class of the considered type of hypersurfaces are given.*

Introduction

The Kähler manifolds with Norden metric have been introduced in [9]. These manifolds form the special class $\mathcal{W}_0$ in the decomposition of the almost complex manifolds with Norden metric, given in [2]. This most important class is contained in each of the basic classes in the mentioned classification.

The natural analogue of the almost complex manifolds with Norden metric in the odd dimensional case are the almost contact manifolds with Norden metric, classified in [4].

In [5] two types of hypersurfaces of an almost complex manifold with Norden metric are constructed as almost contact manifolds with Norden metric, and the class of these hypersurfaces of a $\mathcal{W}_0$-manifold is determined.

An important problem in the differential geometry of the Kähler manifolds with Norden metric is the studying of the manifolds of constant totally real sectional curvatures [3].

*2000 *Mathematics Subject Classification*: Primary 53C15, 53C40; Secondary 32Q60, 53C50. *Keywords*: Hypersurface, curvature, almost contact manifold, Norden metric.

In this paper we study some curvature properties of the real time-like hypersurfaces of Kähler manifolds with Norden metric of constant totally real sectional curvatures and particularly curvature properties of nondegenerate special sections.

1. Preliminaries

1.1. *Almost complex manifolds with Norden metric*

Let (M', J, g') be a $2n'$-dimensional almost complex manifold with Norden metric, i.e. J is an almost complex structure and g' is a metric on M' such that:

$$J^2X = -X, \qquad g'(JX, JY) = -g'(X, Y)$$

for all vector fields $X, Y \in \mathfrak{X}(M')$ (the Lie algebra of the differentiable vector fields on M'). The associated metric $\widetilde{g'}$ of the manifold is given by $\widetilde{g'}(X,Y) = g'(X,JY)$. Both metrics are necessarily of signature (n',n').

Further, X, Y, Z, U will stand for arbitrary differentiable vector fields on the manifold, and x, y, z, u – arbitrary vectors in its tangent space at an arbitrary point.

The (0,3)-tensor F' on M' is defined by $F'(X,Y,Z) = g'((\nabla'_X J)Y,Z)$, where ∇' is the Levi-Civita connection of g'.

A decomposition to three basic classes of the considered manifolds with respect to F' is given in [2]. In this paper we shall consider only the class $\mathcal{W}_0 : \; F' = 0$ of the Kähler manifolds with Norden metric. The complex structure J is parallel on every $\mathcal{W}_0$-manifold, i.e. $\nabla' J = 0$.

The curvature tensor field R', defined by $R'(X,Y)Z = \nabla'_X\nabla'_Y Z - \nabla'_Y\nabla'_X Z - \nabla'_{[X,Y]}Z$, has the property $R'(X, Y, Z, U) = -R'(X, Y, JZ, JU)$ on a $\mathcal{W}_0$-manifold. Using the first Bianchi identity and the last property of R it follows $R'(X, JY, JZ, U) = -R'(X, Y, Z, U)$. Therefore, the tensor field $\widetilde{R'} : \widetilde{R'}(X, Y, Z, U) = R'(X, Y, Z, JU)$ has the properties of a Kähler curvature tensor and it is called *an associated curvature tensor*.

The essential curvature-like tensors are defined by:

$$\begin{aligned}
\pi'_1(x, y, z, u) &= g'(y, z)g'(x, u) - g'(x, z)g'(y, u),\\
\pi'_2(x, y, z, u) &= g'(y, Jz)g'(x, Ju) - g'(x, Jz)g'(y, Ju),\\
\pi'_3(x, y, z, u) &= -g'(y, z)g'(x, Ju) + g'(x, z)g'(y, Ju)\\
&\quad - g'(y, Jz)g'(x, u) + g'(x, Jz)g'(y, u).
\end{aligned}$$

For every nondegenerate section α' in $T_{p'}M'$, $p' \in M'$, with a basis $\{x, y\}$ there are known the following sectional curvatures [1]:

$k'(\alpha';p') = k'(x,y) = \frac{R'(x,y,y,x)}{\pi_1'(x,y,y,x)}$ – the usual Riemannian sectional curvature; $\widetilde{k}'(\alpha';p') = \widetilde{k}'(x,y) = \frac{\widetilde{R}'(x,y,y,x)}{\pi_1'(x,y,y,x)}$ – an associated sectional curvature.

The sectional curvatures of an arbitrary holomorphic section α' (i.e. $J\alpha' = \alpha'$) is zero on a Kähler manifold with Norden metric [1].

For the totally real sections α'(i.e. $J\alpha' \perp \alpha'$) it is proved the following

Theorem 1.1. ([1]) *Let M' ($2n' \geq 4$) be a Kähler manifold with Norden metric. M' is of constant totally real sectional curvatures ν' and $\widetilde{\nu}'$, i.e. $k'(\alpha';p') = \nu'(p')$, $\widetilde{k}'(\alpha';p') = \widetilde{\nu}'(p')$ whenever α' is a nondegenerate totally real section in $T_{p'}M'$, $p' \in M'$, if and only if*

$$R' = \nu' \left[\pi_1' - \pi_2'\right] + \widetilde{\nu}'\pi_3'.$$

Both functions ν' and $\widetilde{\nu}'$ are constant if M' is connected and $2n' \geq 6$.

1.2. *Almost contact manifolds with Norden metric*

Let (M,φ,ξ,η,g) be a $(2n{+}1)$-dimensional almost contact manifold with Norden metric, i.e. (φ,ξ,η) is an almost contact structure determined by a tensor field φ of type $(1,1)$, a vector field ξ and an 1-form η on M satisfying the conditions:

$$\varphi^2 X = -X + \eta(X)\xi, \qquad \eta(\xi) = 1,$$

and in addition the almost contact manifold (M,φ,ξ,η) admits a metric g such that [4]

$$g(\varphi X,\varphi Y) = -g(X,Y) + \eta(X)\eta(Y).$$

There are valid the following immediate corollaries: $\eta\circ\varphi = 0$, $\varphi\xi = 0$, $\eta(X) = g(X,\xi)$, $g(\varphi X,Y) = g(X,\varphi Y)$.

The associated metric $\widetilde{g}$ given by $\widetilde{g}(X,Y) = g(X,\varphi Y) + \eta(X)\eta(Y)$ is a Norden metric, too. Both metrics are indefinite of signature $(n,n+1)$.

The Levi-Civita connection of g will be denoted by ∇. The tensor field F of type (0,3) on M is defined by $F(X,Y,Z) = g((\nabla_X\,\varphi)Y,Z)$.

If $\{e_i,\xi\}\,(i = 1,2,\ldots,2n)$ is a basis of T_pM and (g^{ij}) is the inverse matrix of (g_{ij}), then the following 1-forms are associated with F:

$$\theta(\cdot) = g^{ij}F(e_i,e_j,\cdot), \quad \theta^*(\cdot) = g^{ij}F(e_i,\varphi e_j,\cdot), \quad \omega(\cdot) = F(\xi,\xi,\cdot).$$

A classification of the almost contact manifolds with Norden metric with respect to F is given in [4], where eleven basic classes $\mathcal{F}_i$ are defined. In the present paper we consider the following classes:

$$\begin{aligned}
&\mathcal{F}_4 : F(x,y,z) = -\frac{\theta(\xi)}{2n}\{g(\varphi x,\varphi y)\eta(z)+g(\varphi x,\varphi z)\eta(y)\};\\
&\mathcal{F}_5 : F(x,y,z) = -\frac{\theta^*(\xi)}{2n}\{g(x,\varphi y)\eta(z)+g(x,\varphi z)\eta(y)\};\\
&\mathcal{F}_6 : F(x,y,z) = F(x,y,\xi)\eta(z)+F(x,\xi,z)\eta(y), \quad \theta(\xi)=\theta^*(\xi)=0,\\
&\qquad F(x,y,\xi)=F(y,x,\xi), \quad F(\varphi x,\varphi y,\xi)=-F(x,y,\xi);\\
&\mathcal{F}_{11} : F(x,y,z) = \eta(x)\{\eta(y)\omega(z)+\eta(z)\omega(y)\}.
\end{aligned} \tag{1}$$

The classes $\mathcal{F}_i \oplus \mathcal{F}_j$, etc., are defined in a natural way by the conditions of the basic classes. The special class $\mathcal{F}_0 : F = 0$ is contained in each of the defined classes. The $\mathcal{F}_i^0$-manifold is an $\mathcal{F}_i$ -manifold $(i = 1,4,5,11)$ with closed 1-forms θ, θ^* and $\omega \circ \varphi$.

The following tensors are essential curvature tensors on M:

$$\begin{aligned}
\pi_1(x,y,z,u) &= g(y,z)g(x,u) - g(x,z)g(y,u),\\
\pi_2(x,y,z,u) &= g(y,\varphi z)g(x,\varphi u) - g(x,\varphi z)g(y,\varphi u),\\
\pi_3(x,y,z,u) &= -g(y,z)g(x,\varphi u) + g(x,z)g(y,\varphi u)\\
&\quad - g(y,\varphi z)g(x,u) + g(x,\varphi z)g(y,u),\\
\pi_4(x,y,z,u) &= \eta(y)\eta(z)g(x,u) - \eta(x)\eta(z)g(y,u)\\
&\quad + \eta(x)\eta(u)g(y,z) - \eta(y)\eta(u)g(x,z),\\
\pi_5(x,y,z,u) &= \eta(y)\eta(z)g(x,\varphi u) - \eta(x)\eta(z)g(y,\varphi u)\\
&\quad + \eta(x)\eta(u)g(y,\varphi z) - \eta(y)\eta(u)g(x,\varphi z).
\end{aligned}$$

In [7] it is established that the tensors $\pi_1 - \pi_2 - \pi_4$ and $\pi_3 + \pi_5$ are Kählerian, i.e. they have the condition of a curvature-like tensor L: $L(X,Y,Z,U) = -L(X,Y,\varphi Z,\varphi U)$.

Let R be the curvature tensor of ∇. The tensors R and $\widetilde{R} : \widetilde{R}(x,y,z,u) = R(x,y,z,\varphi u)$ are Kählerian on any $\mathcal{F}_0$-manifold.

There are known the following sectional curvatures with respect to g and R for every nondegenerate section α in T_pM with a basis $\{x,y\}$:

$$k(\alpha;p) = k(x,y) = \frac{R(x,y,y,x)}{\pi_1(x,y,y,x)}, \quad \widetilde{k}(\alpha;p) = \widetilde{k}(x,y) = \frac{\widetilde{R}(x,y,y,x)}{\pi_1(x,y,y,x)}.$$

In [8] there are introduced the following special sections in T_pM: a ξ-section (e.g. $\{\xi,x\}$), a φ-holomorphic section (i.e. $\alpha = \varphi\alpha$) and a totally real section (i.e. $\alpha \perp \varphi\alpha$).

The canonical curvature tensor K is introduced in [7]. The tensor K is a curvature tensor with respect to the canonical connection D defined by

$$D_X Y = \nabla_X Y + \frac{1}{2}\left\{(\nabla_X \varphi)\varphi Y + (\nabla_X \eta)Y.\xi\right\} - \eta(Y)\nabla_X \xi. \qquad (2)$$

The connection D is a natural connection, i.e. the structural tensors are parallel with respect to D. Let us note that the tensor K out of $\mathcal{F}_0$ has the properties of R in $\mathcal{F}_0$.

2. Curvatures on the real time-like hypersurfaces of a Kähler manifold with Norden metric

In [5] two types of real hypersurfaces of a complex manifold with Norden metric are introduced. The obtained submanifolds are almost contact manifolds with Norden metric. Let us recall the real time-like hypersurface with respect to the Norden metric.

The hypersurface M of an almost complex manifold with Norden metric (M', J, g'), determined by the condition the normal unit N to be time-like regarding g' (i.e. $g'(N,N) = -1$), equipped with the almost contact structure with Norden metric

$$\begin{aligned} &\varphi := J + \cos t.g'(\cdot, JN)\{\cos t.N - \sin t.JN\},\\ &\xi := \sin t.N + \cos t.JN, \quad \eta := \cos t.g'(\cdot, JN), \quad g := g'|_M, \end{aligned} \qquad (3)$$

where $t := \arctan\{g'(N, JN)\}$ for $t \in \left(-\frac{\pi}{2}; \frac{\pi}{2}\right)$, is called *a real time-like hypersurface* of (M', J, g').

In the case when (M', J, g') is a Kähler manifold with Norden metric (i.e. a $\mathcal{W}_0$-manifold), in [6] it is ascertained the following statement: The class $\mathcal{F}_4 \otimes \mathcal{F}_5 \otimes \mathcal{F}_6 \otimes \mathcal{F}_{11}$ is the class of the real time-like hypersurfaces of a Kähler manifold with Norden metric. There are 16 classes of these hypersurfaces in all. When $n = 1$ the class $\mathcal{F}_6$ is restricted to $\mathcal{F}_0$. Therefore, for a 4-dimensional Kähler manifold with Norden metric there are only 8 classes of the considered hypersurfaces.

The tensor F and the second fundamental tensor A of the considered type of hypersurfaces have the following form, respectively:

$$\begin{aligned} &F(X,Y,Z) = \sin t\left\{g(AX,\varphi Y)\eta(Z) + g(AX,\varphi Z)\eta(Y)\right\}\\ &- \cos t\left\{g(AX,Y)\eta(Z) + g(AX,Z)\eta(Y) - 2\eta(AX)\eta(Y)\eta(Z)\right\},\\ &AX = -\tfrac{dt(\xi)}{2\cos t}\eta(X)\xi - \sin t\{\nabla_X \xi + g(\nabla_\xi \xi, X)\xi\}\\ &\qquad + \cos t\{\varphi\nabla_X \xi + g(\varphi\nabla_\xi \xi, X)\xi\}. \end{aligned} \qquad (4)$$

The basic classes of the considered hypersurfaces are characterized in terms of the second fundamental tensor by the conditions [5]:

$$\begin{aligned}
\mathcal{F}_0 &: A=-\tfrac{dt(\xi)}{2\cos t}\eta\otimes\xi;\\
\mathcal{F}_4 &: A=-\tfrac{dt(\xi)}{2\cos t}\eta\otimes\xi-\tfrac{\theta(\xi)}{2n}\{\sin t.\varphi-\cos t.\varphi^2\};\\
\mathcal{F}_5 &: A=-\tfrac{dt(\xi)}{2\cos t}\eta\otimes\xi+\tfrac{\theta^*(\xi)}{2n}\{\cos t.\varphi+\sin t.\varphi^2\};\\
\mathcal{F}_6 &: A\circ\varphi=\varphi\circ A\,,\ \ trA-\tfrac{dt(\xi)}{2\cos t}=tr(A\circ\varphi)=0;\\
\mathcal{F}_{11} &: A=-\tfrac{dt(\xi)}{2\cos t}\eta\otimes\xi-\cos t\{\eta\otimes\Omega+\omega\otimes\xi\}\\
&\qquad\qquad -\sin t\{\eta\otimes\varphi\Omega+(\omega\circ\varphi)\otimes\xi\},\quad \omega(\cdot)=g(\cdot,\Omega).
\end{aligned}$$

According to the formulas of Gauss and Weingarten in this case $\nabla'_X Y = \nabla_X Y - g(AX,Y)N$, $\nabla'_X N = -AX$, we get the relation between the curvature tensors R' and R of the $\mathcal{W}_0$-manifold (M',J,g') and its hypersurface (M,φ,ξ,η,g), respectively:

$$\begin{aligned}
R'(x,y,z,u) &= R(x,y,z,u)+\pi_1(Ax,Ay,z,u),\\
R'(x,y)N &= -\left(\nabla_x A\right)y+\left(\nabla_y A\right)x.
\end{aligned}$$

Hence, having in mind Theorem 1.1, we obtain:

$$\begin{aligned}
R(x,y,z,u) &= \left\{\nu'\left[\pi'_1-\pi'_2\right]+\widetilde{\nu}'\pi'_3\right\}(x,y,z,u)-\pi_1(Ax,Ay,z,u),\\
R(x,y,\varphi z,\varphi u) &= -\left\{R-\nu'[\pi_4-\tan t\pi_5]+\widetilde{\nu}'[\pi_5+\tan t\pi_4]\right\}(x,y,z,u)\\
&\qquad -[\pi_1+\pi_2](Ax,Ay,z,u),\\
R(x,y)\xi &= \left\{\nu'[\pi_4-\tan t\pi_5]-\widetilde{\nu}'[\pi_5+\tan t\pi_4]\right\}(x,y)\xi-\pi_1(Ax,Ay)\xi,\\
R(x,y)N &= -\tfrac{1}{\cos t}[\nu'\pi_5+\widetilde{\nu}'\pi_4](x,y)\xi.
\end{aligned}$$

Therefore

$$(\nabla_x A)y-(\nabla_y A)x=\frac{1}{\cos t}[\nu'\pi_5+\widetilde{\nu}'\pi_4](x,y)\xi. \tag{5}$$

Having in mind the equations: $g'(y,Jz)=g(y,\varphi z)+\tan t\ \eta(y)\eta(z)$, $\pi'_1=\pi_1$, $\pi'_2=\pi_2+\tan t\ \pi_5$, $\pi'_3=\pi_3-\tan t\ \pi_4$, which are valid for real time-like hypersurfaces, we obtain

Proposition 2.1. *A real time-like hypersurface of a Kähler manifold with Norden metric of constant totally real sectional curvatures* ν' *and* $\widetilde{\nu}'$ *has* *the following curvature properties:*

$$R(x,y,z,u) = \left\{\nu'[\pi_1 - \pi_2 - \tan t\pi_5] + \widetilde{\nu'}[\pi_3 - \tan t\pi_4]\right\}(x,y,z,u)$$
$$-\pi_1(Ax,Ay,z,u),$$
$$\tau = 4n^2\nu' - 4n\widetilde{\nu'}\tan t - (\mathrm{tr}A)^2 + \mathrm{tr}A^2,$$
$$\widetilde{\tau} = -2n\nu'\tan t + 2n(2n-1)\widetilde{\nu'} - \mathrm{tr}A\mathrm{tr}(A\circ\varphi) + \mathrm{tr}(A^2\circ\varphi);$$

for a ξ-section $\{\xi,x\}$

$$k(\xi,x) = \nu' - \widetilde{\nu'}\tan t - [\nu'\tan t + \widetilde{\nu'}]\frac{g(x,\varphi x)}{g(x,x)-\eta(x)^2} - \frac{\pi_1(A\xi,Ax,x,\xi)}{g(x,x)-\eta(x)^2};$$

for a φ-holomorphic section $\left\{\varphi x,\varphi^2 x\right\}$ and for a totally real section $\{x,y\}$, orthogonal to ξ, respectively:

$$k(\varphi x,\varphi^2 x) = -\frac{\pi_1(A\varphi x,A\varphi^2 x,\varphi^2 x,\varphi x)}{\pi_1(\varphi x,\varphi^2 x,\varphi^2 x,\varphi x)},\quad k(x,y) = \nu' - \frac{\pi_1(Ax,Ay,y,x)}{\pi_1(x,y,y,x)}.$$

If (M,φ,ξ,η,g) is a real time-like hypersurface of $\mathcal{W}_0$-manifold, then (2), (4) and (5) imply that the canonical curvature tensor has the form

$$K(x,y,z,u) = R(x,y,\varphi^2 z,\varphi^2 u) + \pi_1(Ax,Ay,\varphi z,\varphi u)$$
$$+\sin t\left\{\sin t[\pi_1 - \pi_2 - \pi_4] - \cos t[\pi_3 + \pi_5]\right\}(Ax,Ay,z,u).$$

Then, because of the last equation and Proposition 2.1 we have

Proposition 2.2. *Let (M,φ,ξ,η,g) be a real time-like hypersurface of a $\mathcal{W}_0$-manifold (M',J,g') of constant totally real sectional curvatures. Then K of M is Kählerian and*

$$K(x,y,z,u) = \left\{\nu'[\pi_1 - \pi_2 - \pi_4] + \widetilde{\nu'}[\pi_3 + \pi_5]\right\}(x,y,z,u)$$
$$-\cos t\left\{\cos t[\pi_1 - \pi_2 - \pi_4] + \sin t[\pi_3 + \pi_5]\right\}(Ax,Ay,z,u),$$
$$\tau(K) = 4n(n-1)\nu' - \cos t(a\cos t + 2b\sin t),$$
$$\widetilde{\tau}(K) = 4n(n-1)\widetilde{\nu'} - \cos t(a\sin t - 2b\cos t),$$
$$a = (\mathrm{tr}A)^2 - \mathrm{tr}A^2 - [\mathrm{tr}(A\circ\varphi)]^2 + \mathrm{tr}(A\circ\varphi)^2 - 2\eta(A\xi)\mathrm{tr}A + 2g(A\xi,A\xi),$$
$$b = \mathrm{tr}(A^2\circ\varphi) - \mathrm{tr}A\mathrm{tr}(A\circ\varphi) + \eta(A\xi)\mathrm{tr}(A\circ\varphi) - g(\varphi A\xi,A\xi).$$

3. Curvatures on $\mathcal{W}_0$'s real time-like hypersurfaces, belonging to the main classes

Now, let (M,φ,ξ,η,g) belong to the widest integrable main class $\mathcal{F}_4\oplus\mathcal{F}_5$ of the real time-like hypersurfaces. Let us recall that a class of almost contact manifolds with Norden metric is said to be main if the tensor F is expressed explicitly by the structural tensors φ, ξ, η, g. In this case for the second fundamental tensor we have [5]:

$$\begin{aligned} A &= -\tfrac{dt(\xi)}{2\cos t}\eta\otimes\xi - \tfrac{1}{2n}\big\{[\theta(\xi)\sin t - \theta^*(\xi)\cos t]\varphi \\ &\quad - [\theta(\xi)\cos t + \theta^*(\xi)\sin t]\,\varphi^2\big\}, \\ \mathrm{tr}A &= -\tfrac{dt(\xi)}{2\cos t} - \theta(\xi)\cos t - \theta^*(\xi)\sin t, \\ \mathrm{tr}(A\circ\varphi) &= \theta(\xi)\sin t - \theta^*(\xi)\cos t. \end{aligned} \tag{6}$$

Then, having in mind the last identities and Proposition 2.1, we obtain

Corollary 3.1. *If a real time-like hypersurface of a Kähler manifold with Norden metric of constant totally real sectional curvatures is an $\mathcal{F}_4\oplus\mathcal{F}_5$-manifold, then it has the following curvature properties:*

$$\begin{aligned} R &= \nu'[\pi_1 - \pi_2 - \tan t\pi_5] + \widetilde{\nu'}[\pi_3 - \tan t\pi_4] \\ &\quad -\tfrac{dt(\xi)}{4n\cos t}\big\{\theta(\xi)[\sin t\pi_5 + \cos t\pi_4] + \theta^*(\xi)[\sin t\pi_4 - \cos t\pi_5]\big\} \\ &\quad -\tfrac{\theta(\xi)^2+\theta^*(\xi)^2}{4n^2}\pi_2 - \tfrac{\left(\theta(\xi)\cos t+\theta^*(\xi)\sin t\right)^2}{4n^2}[\pi_1-\pi_2-\pi_4] \\ &\quad +\tfrac{\left(\theta(\xi)\cos t+\theta^*(\xi)\sin t\right)\left(\theta(\xi)\sin t-\theta^*(\xi)\cos t\right)}{4n^2}[\pi_3+\pi_5], \\ \tau &= 4n(n\nu' - \widetilde{\nu'}\tan t) - dt(\xi)\theta(\xi) - dt(\xi)\theta^*(\xi)\tan t \\ &\quad -\tfrac{n-1}{n}(\theta(\xi)\cos t + \theta^*(\xi)\sin t)^2 - \tfrac{\theta(\xi)^2+\theta^*(\xi)^2}{2n}, \\ \widetilde{\tau} &= 2n(n-1)\widetilde{\nu'} + 2n\nu'\tan t + \tfrac{dt(\xi)\theta(\xi)}{2}\tan t - \tfrac{dt(\xi)\theta^*(\xi)}{2} \\ &\quad +\tfrac{n-1}{n}\big(\theta(\xi)\sin t - \theta^*(\xi)\cos t)(\theta(\xi)\cos t + \theta^*(\xi)\sin t\big), \\ k(\varphi x,\varphi^2 x) &= -\tfrac{\theta(\xi)^2+\theta^*(\xi)^2}{4n^2}, \quad k(x,y) = \nu' - \tfrac{\left(\theta(\xi)\cos t+\theta^*(\xi)\sin t\right)^2}{4n^2}. \end{aligned}$$

Let us remark that we can obtain the corresponding properties for the classes $\mathcal{F}_4$, $\mathcal{F}_5$ and $\mathcal{F}_0$, if we substitute $\theta^*(\xi)=0$, $\theta(\xi)=0$ and $\theta(\xi)=\theta^*(\xi)=0$, respectively.

Using the equations (2) and (1), we express the canonical connection explicitly for the class $\mathcal{F}_4\oplus\mathcal{F}_5$ as follows

$$D_XY = \nabla_XY + \frac{\theta(\xi)}{2n}\{g(x,\varphi y)\xi - \eta(y)\varphi x\} - \frac{\theta^*(\xi)}{2n}\left\{g(\varphi x,\varphi y)\xi - \eta(y)\varphi^2 x\right\}.$$

Let $(M,\varphi,\xi,\eta,g)\in\mathcal{F}_4^0\oplus\mathcal{F}_5^0$, i.e. (M,φ,ξ,η,g) is an $(\mathcal{F}_4\oplus\mathcal{F}_5)$-manifold with closed 1-forms θ and θ^*. The canonical curvature tensor K of any $\left(\mathcal{F}_4^0\oplus\mathcal{F}_5^0\right)$-manifold is Kählerian and it has the form

$$\begin{aligned} K &= R + \tfrac{\xi\theta(\xi)}{2n}\pi_5 + \tfrac{\xi\theta^*(\xi)}{2n}\pi_4 \\ &\quad + \tfrac{\theta^2(\xi)}{4n^2}[\pi_2 - \pi_4] + \tfrac{\theta^{*2}(\xi)}{4n^2}\pi_1 - \tfrac{\theta(\xi)\theta^*(\xi)}{4n^2}[\pi_3 - \pi_5]. \end{aligned}$$

Then, using Corollary 3.1, we ascertain the truthfulness of the following

Corollary 3.2. *If a real time-like hypersurface of a Kähler manifold with Norden metric of constant totally real sectional curvatures is an* $\left(\mathcal{F}_4^0 \oplus \mathcal{F}_5^0\right)$*-manifold, then* K *is expressed in the following way:*

$$\begin{aligned}
K = &\left(\nu' + \tfrac{\theta^*(\xi)}{4n^2}\right)[\pi_1 - \pi_2] + \left(\widetilde{\nu}' - \tfrac{\theta(\xi)\theta^*(\xi)}{4n^2}\right)\pi_3 \\
&-\left(\widetilde{\nu}' \tan t + \tfrac{dt(\xi)\theta(\xi)}{4n} + \tfrac{dt(\xi)\theta^*(\xi)}{4n}\tan t - \tfrac{\xi\theta^*(\xi)}{2n} + \tfrac{\theta(\xi)^2}{4n^2}\right)\pi_4 \\
&-\left(\nu' \tan t + \tfrac{dt(\xi)\theta(\xi)}{4n}\tan t - \tfrac{dt(\xi)\theta^*(\xi)}{4n} - \tfrac{\xi\theta(\xi)}{2n} - \tfrac{\theta(\xi)\theta^*(\xi)}{4n^2}\right)\pi_5 \\
&-\tfrac{\left(\theta(\xi)\cos t + \theta^*(\xi)\sin t\right)^2}{4n^2}[\pi_1 - \pi_2 - \pi_4] \\
&+\tfrac{\left(\theta(\xi)\sin t - \theta^*(\xi)\cos t\right)\left(\theta(\xi)\cos t + \theta^*(\xi)\sin t\right)}{4n^2}[\pi_3 + \pi_5].
\end{aligned}$$

We compute the expression $\left(\nabla_x A\right) y - \left(\nabla_y A\right) x$ using (6) and we compare the result with (5). Thus, we get the relations

$$\begin{aligned}
\nu' = &-\tfrac{dt(\xi)\theta(\xi)}{4n} + \tfrac{\cos t}{2n}\left[\xi\theta(\xi)\sin t - \xi\theta^*(\xi)\cos t\right] \\
&+\tfrac{\cos^2 t}{4n^2}\left[\theta(\xi)^2 - \theta^*(\xi)^2\right] + \tfrac{\sin t\cos t}{2n^2}\theta(\xi)\theta^*(\xi) \\
&+\tfrac{dt(\xi)}{2n}\cos t\left[\theta(\xi)\cos t + \theta^*(\xi)\sin t\right], \\
\widetilde{\nu}' = &-\tfrac{dt(\xi)\theta^*(\xi)}{4n} + \tfrac{\cos t}{2n}\left[\xi\theta(\xi)\cos t + \xi\theta^*(\xi)\sin t\right] \\
&+\tfrac{\sin t\cos t}{4n^2}\left[\theta^*(\xi)^2 - \theta(\xi)^2\right] + \tfrac{\cos^2 t}{2n^2}\theta(\xi)\theta^*(\xi) \\
&-\tfrac{dt(\xi)}{2n}\cos t\left[\theta(\xi)\sin t - \theta^*(\xi)\cos t\right].
\end{aligned} \tag{7}$$

Hence, we have

$$\begin{aligned}
K &= \lambda[\pi_1 - \pi_2 - \pi_4] + \mu[\pi_3 + \pi_5], \\
R &= \lambda[\pi_1 - \pi_2 - \pi_4] + \mu[\pi_3 + \pi_5] - \tfrac{\xi\theta^*(\xi)}{2n}\pi_4 - \tfrac{\xi\theta(\xi)}{2n}\pi_5 \\
&\quad - \tfrac{\theta^*(\xi)^2}{4n^2}\pi_1 - \tfrac{\theta(\xi)^2}{4n^2}[\pi_2 - \pi_4] + \tfrac{\theta(\xi)\theta^*(\xi)}{4n^2}[\pi_3 - \pi_5], \\
\lambda &= -\tfrac{dt(\xi)\theta(\xi)}{4n} + \tfrac{dt(\xi)}{2n}\cos t\left[\theta(\xi)\cos t + \theta^*(\xi)\sin t\right] \\
&\quad + \tfrac{\cos t}{2n}\left[\xi\theta(\xi)\sin t - \xi\theta^*(\xi)\cos t\right], \\
\mu &= -\tfrac{dt(\xi)\theta^*(\xi)}{4n} - \tfrac{dt(\xi)}{2n}\cos t\left[\theta(\xi)\sin t - \theta^*(\xi)\cos t\right] \\
&\quad + \tfrac{\cos t}{2n}\left[\xi\theta(\xi)\cos t + \xi\theta^*(\xi)\sin t\right].
\end{aligned}$$

We solve the system (7) with respect to the functions $\theta(\xi)$ and $\theta^*(\xi)$ for $t = \mathrm{const}$ and get

$$\theta(\xi) = 2\varepsilon n\sqrt{\tfrac{\nu'\cos t - \widetilde{\nu}'\sin t + \sqrt{\nu'^2 + \widetilde{\nu}'^2}}{2\cos t}}, \theta^*(\xi) = 2\varepsilon n\tfrac{\sqrt{\cos t}\left(\nu'\tan t + \widetilde{\nu}'\right)}{\sqrt{2\left(\nu'\cos t - \widetilde{\nu}'\sin t + \sqrt{\nu'^2 + \widetilde{\nu}'^2}\right)}},$$

where $\varepsilon = \pm 1$. Since ν' and $\widetilde{\nu'}$ are pointwise constant for M'^4 $(n = 1)$ and they are absolute constants for M'^{2n+2} $(n \geq 2)$ (Theorem 1.1), then the functions $\theta(\xi)$ and $\theta^*(\xi)$, which determine the real time-like hypersurface as an almost contact manifold with Norden metric, are also pointwise constant on M^3 and absolute constants on M^{2n+1} $(n \geq 2)$. Hence, we have

Theorem 3.1. *Let (M', J, g') be a Kähler manifold with Norden metric of constant totally real sectional curvatures. Let the $\left(\mathcal{F}_4^0 \oplus \mathcal{F}_5^0\right)$-manifold $(M, \varphi, \xi, \eta, g)$, $\dim M \geq 5$, be its real time-like hypersurface, defined by (3). If $t = \mathrm{const}$, then $K = 0$ on M and*

$$
\begin{aligned}
&R = -\tfrac{\theta(\xi)^2}{4n^2}\left[\pi_2 - \pi_4\right] - \tfrac{\theta^*(\xi)^2}{4n^2}\pi_1 + \tfrac{\theta(\xi)\theta^*(\xi)}{4n^2}\left[\pi_3 - \pi_5\right],\\
&\tau = \tfrac{\theta(\xi)^2}{2n} - (2n+1)\tfrac{\theta^*(\xi)^2}{2n}, \qquad \widetilde{\tau} = \tfrac{\theta(\xi)\theta^*(\xi)}{2n},\\
&k(\xi, x) = \tfrac{\theta(\xi)^2 - \theta^*(\xi)^2}{4n^2} + \tfrac{2\theta(\xi)\theta^*(\xi)}{4n^2}\tfrac{g(x,\varphi x)}{g(\varphi x, \varphi x)},\\
&k\left(\varphi x, \varphi^2 x\right) = -\tfrac{\theta(\xi)^2 + \theta^*(\xi)^2}{4n^2}, \qquad k(x, y) = -\tfrac{\theta^*(\xi)^2}{4n^2}.
\end{aligned}
$$

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