Lie groups as four-dimensional complex manifolds with Norden metric

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Abstract

Two examples of 4-dimensional complex manifolds with Norden metric are constructed by means of Lie groups and Lie algebras. Both manifolds are characterized geometrically. The form of the curvature tensor for each of the examples is obtained. Conditions these manifolds to be isotropic-Kählerian are given. [1] [2]

Introduction

Almost complex manifolds with Norden metric are introduced in [9] as generalized B-manifolds. These manifolds are classified into eight classes in [4], and equivalent characteristic conditions for each of these classes are obtained in [5].

Examples of the basic classes $\mathcal{W}_0$, $\mathcal{W}_1$ and $\mathcal{W}_2$ of the integrable almost complex manifolds with Norden metric are given in [2]. An example of the only basic class $\mathcal{W}_3$ with a non-integrable almost complex structure is given in [10].

In this paper our purpose is to construct examples of two classes of integrable almost complex manifolds with Norden metric, namely the class of the Kähler manifolds with Norden metric and the class of the complex manifolds with Norden metric. Both examples are 4-dimensional manifolds and they are obtained by constructing 4-parametric families of Lie algebras corresponding to real connected Lie groups. The manifolds obtained in this way are characterized geometrically. In particular, we find the form of the curvature tensor for each of the examples and we study the conditions these manifolds to be isotropic-Kählerian.

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[2]*Keywords and phrases*: almost complex manifold, Norden metric, indefinite metric, bi-invariant complex structure, abelian complex structure, complex manifold, Kähler manifold.

1 Almost complex manifolds with Norden metric

1.1 Preliminaries

Let (M, J, g) be a $2n$-dimensional almost complex manifold with Norden metric, i.e. J is an almost complex structure and g is a metric on M such that

$$J^2X = -X, \qquad g(JX, JY) = -g(X, Y) \tag{1.1}$$

for all differentiable vector fields X, Y on M, i.e. $X, Y \in \mathfrak{X}(M)$.

The associated metric $\widetilde{g}$ of g given by $\widetilde{g}(X, Y) = g(X, JY)$ is a Norden metric, too. Both metrics are necessarily indefinite of signature (n, n).

Further, X, Y, Z, W $(x, y, z, w$, respectively) will stand for arbitrary differentiable vector fields on M (vectors in T_pM, $p \in M$, respectively).

Let ∇ be the Levi-Civita connection of the metric g. Then, the tensor field F of type $(0, 3)$ on M is defined by

$$F(X, Y, Z) = g\left((\nabla_X J)Y, Z\right). \tag{1.2}$$

It has the following symmetries

$$F(X, Y, Z) = F(X, Z, Y) = F(X, JY, JZ). \tag{1.3}$$

Let $\{e_i\}$ $(i = 1, 2, \dots, 2n)$ be an arbitrary basis of T_pM at a point p of M and let g^{ij} be the components of the inverse matrix of g with respect to this basis.

The Lie form θ associated with F is defined by

$$\theta(z) = g^{ij}F(e_i, e_j, z) \tag{1.4}$$

and the corresponding Lie vector is denoted by Ω, i.e.

$$\theta(z) = g(z, \Omega). \tag{1.5}$$

The Nijenhuis tensor field N given by

$$N(X, Y) = [JX, JY] - [X, Y] - J[JX, Y] - J[X, JY]. \tag{1.6}$$

By means of the covariant derivative of J this tensor is expressed as follows

$$N(X, Y) = (\nabla_X J)JY - (\nabla_Y J)JX + (\nabla_{JX} J)Y - (\nabla_{JY} J)X. \tag{1.7}$$

It is known [11] that the almost complex structure J is complex if it is integrable, i.e. if $N = 0$.

A classification of the almost complex manifolds with Norden metric is introduced in [4], where eight classes of these manifolds are characterized according to the properties of F. The class $\mathcal{W}_0$ of the Kähler manifolds with Norden metric, the three basic classes $\mathcal{W}_1$, $\mathcal{W}_2$, $\mathcal{W}_3$ and the class $\mathcal{W}_1 \oplus \mathcal{W}_2$ of the complex manifolds with Norden metric are given as follows:

$$\begin{aligned}
&\mathcal{W}_0 : F(X,Y,Z) = 0;\\
&\mathcal{W}_1 : F(X,Y,Z) = \tfrac{1}{2n}\left[g(X,Y)\theta(Z) + g(X,Z)\theta(Y)\right.\\
&\qquad\qquad\qquad \left. + g(X,JY)\theta(JZ) + g(X,JZ)\theta(JY)\right];\\
&\mathcal{W}_2 : F(X,Y,JZ) + F(Y,Z,JX) + F(Z,X,JY) = 0, \quad \theta = 0;\\
&\mathcal{W}_3 : F(X,Y,Z) + F(Y,Z,X) + F(Z,X,Y) = 0;\\
&\mathcal{W}_1 \oplus \mathcal{W}_2 : N = 0 \Leftrightarrow F(X,Y,JZ) + F(Y,Z,JX) + F(Z,X,JY) = 0.
\end{aligned} \tag{1.8}$$

Let R be the curvature tensor of ∇, i.e.

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{1.9}$$

The corresponding tensor of type $(0,4)$ is given by

$$R(X,Y,Z,W) = g\left(R(X,Y)Z, W\right). \tag{1.10}$$

The Ricci tensor ρ and the scalar curvatures τ and $\overset{*}{\tau}$ of R are defined by:

$$\rho(y,z) = g^{ij} R(e_i, y, z, e_j), \quad \tau = g^{ij}\rho(e_i, e_j), \quad \overset{*}{\tau} = g^{ij}\rho(e_i, Je_j). \tag{1.11}$$

Definition 1.1 A tensor L of type $(0,4)$ is said to be *curvature-like* if it satisfies the following conditions for any $X,Y,Z,W \in \mathfrak{X}(M)$:

$$\begin{aligned}
&L(X,Y,Z,W) = -L(Y,X,Z,W) = -L(X,Y,W,Z);\\
&L(X,Y,Z,W) + L(Y,Z,X,W) + L(Z,X,Y,W) = 0.
\end{aligned} \tag{1.12}$$

Definition 1.2 A curvature-like tensor L is said to be *Kählerian* if

$$L(X,Y,JZ,JW) = -L(X,Y,Z,W), \quad X,Y,Z,W \in \mathfrak{X}(M). \tag{1.13}$$

Let S be a symmetric and J-hybrid tensor of type $(0,2)$, i.e.

$$S(X,Y) = S(Y,X), \quad S(JX,JY) = -S(X,Y). \tag{1.14}$$

We consider the following curvature-like tensors of type $(0,4)$:

$$\begin{aligned}
&\psi_1(S)(X,Y,Z,W) = g(Y,Z)S(X,W) - g(X,Z)S(Y,W)\\
&\qquad\qquad\qquad + g(X,W)S(Y,Z) - g(Y,W)S(X,Z);\\
&\psi_2(S)(X,Y,Z,W) = g(Y,JZ)S(X,JW) - g(X,JZ)S(Y,JW)\\
&\qquad\qquad\qquad + g(X,JW)S(Y,JZ) - g(Y,JW)S(X,JZ);\\
&\pi_1 = \tfrac{1}{2}\psi_1(g); \qquad \pi_2 = \tfrac{1}{2}\psi_2(g); \qquad \pi_3 = -\psi_1(\widetilde{g}) = \psi_2(\widetilde{g}).
\end{aligned} \tag{1.15}$$

It is well-known that the Weyl tensor W on a $2n$-dimensional pseudo-Riemannian manifold $(2n \geq 4)$ is defined by

$$W = R - \frac{1}{2n-2}\left\{\psi_1(\rho) - \frac{\tau}{2n-1}\pi_1\right\}. \tag{1.16}$$

The Weyl tensor vanishes if and only if the manifold is conformally flat.

Let $\alpha = \{x, y\}$ be a non-degenerate 2-plane spanned by the vectors $x, y \in T_pM$, $p \in M$. Then, the sectional curvatures of α are given by:

$$\nu(\alpha; p) = \frac{R(x,y,y,x)}{\pi_1(x,y,y,x)}, \qquad \overset{*}{\nu}(\alpha; p) = \frac{R(x,y,y,Jx)}{\pi_1(x,y,y,x)}. \tag{1.17}$$

We consider the following basic sectional curvatures in T_pM with respect to the structures J and g:

- *holomorphic sectional curvatures* if $J\alpha = \alpha$;
- *totally real sectional curvatures* if $J\alpha \perp \alpha$ with respect to g.

In [8], *a holomorphic bisectional curvature* $h(x, y)$ for a pair of holomorphic 2-planes $\alpha_1 = \{x, Jx\}$ and $\alpha_2 = \{y, Jy\}$ is defined by

$$h(x,y) = -\frac{R(x,Jx,y,Jy)}{\sqrt{\pi_1(x,Jx,x,Jx)\pi_1(y,Jy,y,Jy)}}, \tag{1.18}$$

where x, y do not lie along the totally isotropic directions, i.e. both couples $(g(x,x),\ g(x,Jx))$ and $(g(y,y),\ g(y,Jy))$ are different from the couple $(0,0)$. The holomorphic bisectional curvature is invariant with respect to the basis of the 2-planes α_1 and α_2. In particular, if $\alpha_1 = \alpha_2$, then the holomorphic bisectional curvature coincides with the holomorphic sectional curvature of the 2-plane $\alpha_1 = \alpha_2$.

Let us note that the square norm $\|\nabla J\|^2$ of ∇J is given by

$$\|\nabla J\|^2 = g^{ij}g^{kl}g\left((\nabla_{e_i}J)e_k, (\nabla_{e_j}J)e_l\right). \tag{1.19}$$

Definition 1.3 [10] An almost complex manifold with Norden metric satisfying the condition $\|\nabla J\|^2 = 0$ is called *an isotropic Kähler manifold with Norden metric.*

It is clear that if a manifold belongs to the class $\mathcal{W}_0$, then it is isotropic Kählerian. The inverse statement is not always true.

1.2 Geometric properties of complex manifolds with Norden metric

It is well-known [13] that the curvature tensor R on any almost complex manifold with Norden metric satisfies the identity

$$(\nabla_X F)(Y,Z,JW) - (\nabla_Y F)(X,Z,JW) = R(X,Y,Z,W) + R(X,Y,JZ,JW). \tag{1.20}$$

Further, we obtain by means of (1.2) and (1.3) the following property of ∇F

$$(\nabla_X F)(Y, JZ, W) = -(\nabla_X F)(Y, Z, JW) - g((\nabla_X J)Z, (\nabla_Y J)W) \\ -g((\nabla_X J)W, (\nabla_Y J)Z). \tag{1.21}$$

Theorem 1.1 *Let (M, J, g) be a complex manifold with Norden metric. Then*

$$\underset{X,Y,Z}{\mathfrak{S}} \{R(JX, JY, Z, W) + R(X, Y, JZ, JW)\} = \\ \underset{X,Y,Z}{\mathfrak{S}} g((\nabla_X J)Y - (\nabla_Y J)X, (\nabla_W J)Z - (\nabla_Z J)W), \tag{1.22}$$

where $\mathfrak{S}$ is the cyclic sum by three arguments.

Proof. Since (M, J, g) belongs to the class $\mathcal{W}_1 \oplus \mathcal{W}_2$ of the complex manifolds with Norden metric, the characteristic condition

$$F(Y, Z, JW) + F(Z, W, JY) + F(W, Y, JZ) = 0 \tag{1.23}$$

holds. Then, by covariant differentiation of (1.23) we obtain

$$(\nabla_X F)(Y, Z, JW) + (\nabla_X F)(Z, W, JY) + (\nabla_X F)(W, Y, JZ) \\ +g((\nabla_X J)W, (\nabla_Y J)Z) + g((\nabla_X J)Y, (\nabla_Z J)W) + g((\nabla_X J)Z, (\nabla_W J)Y) = 0. \tag{1.24}$$

Taking into account (1.20), (1.21), (1.24) we get (1.22) after straightforward computations. ■

Let us remark that the identity (1.22) is given in [7] as a corollary of the main theorem proved there. We established (1.22) by direct computation.

In [8] three basic classes $\mathfrak{L}_i$ $(i = 1, 2, 3)$ are introduced for a curvature-like tensor L on an almost complex manifold with Norden metric:

$$L \in \mathfrak{L}_1 \Leftrightarrow L(X, Y, JZ, JW) = -L(X, Y, Z, W), \text{ i.e. } L \text{ is a Kähler tensor;} \\ L \in \mathfrak{L}_2 \Leftrightarrow L(X, Y, JZ, JW) + L(Y, Z, JX, JW) + L(Z, X, JY, JW) = 0; \\ L \in \mathfrak{L}_3 \Leftrightarrow L(JX, JY, JZ, JW) = L(X, Y, Z, W). \tag{1.25}$$

Definition 1.4 A curvature-like tensor L on an almost complex manifold with Norden metric is said to be in the class $\mathfrak{L}_1'$ if it is invariant with respect to J, i.e.

$$L(X, Y, JZ, JW) = L(X, Y, Z, W), \quad X, Y, Z, W \in \mathfrak{X}(M). \tag{1.26}$$

It is clear that we have the following inclusion relations between the above considered classes:

$$\begin{matrix} \mathfrak{L}_1 \\ \mathfrak{L}_1' \end{matrix} \subset \mathfrak{L}_2 \subset \mathfrak{L}_3. \tag{1.27}$$

By (1.1), (1.11), (1.14), (1.25) and (1.26) it is easy to prove the following

Proposition 1.2 *Let R be the curvature tensor on an almost complex manifold with Norden metric. Then the following implications hold:*

(i) *If R is in $\mathfrak{L}_3$, then its Ricci tensor ρ is J-hybrid, i.e. $\rho(JX, JY) = -\rho(X, Y)$;*

(ii) *If R is in $\mathfrak{L}'_1$, then $\overset{*}{\tau} = 0$.*

Next, Theorem 1.1 and Definition 1.4 imply

Corollary 1.3 *Let (M, J, g) be a complex manifold with Norden metric and let R belong to the class $\mathfrak{L}'_1$. Then, we have*

$$\underset{X,Y,Z}{\mathfrak{S}} g\left((\nabla_X J)Y - (\nabla_Y J)X, (\nabla_Z J)W - (\nabla_W J)Z\right) = 0. \qquad (1.28)$$

Further, let us denote

$$K(X, Y, Z, W) = g\left((\nabla_X J)Y - (\nabla_Y J)X, (\nabla_Z J)W - (\nabla_W J)Z\right). \qquad (1.29)$$

Then, (1.29) implies

$$K(X, Y, Z, W) = -K(Y, X, Z, W) = -K(X, Y, W, Z). \qquad (1.30)$$

By (1.30), Corollary 1.3 and Definition 1.1 we establish that K is a curvature-like tensor on any complex manifold with Norden metric if the curvature tensor R is in $\mathfrak{L}'_1$. Moreover, by (1.7) and $N = 0$, it is easy to prove that

$$K(X, Y, JZ, JW) = K(X, Y, Z, W), \qquad (1.31)$$

i.e. the tensor K belongs to the class $\mathfrak{L}'_1$, too.

2 A Lie group as a 4-dimensional Kähler manifold with Norden metric

Let V be a 4-dimensional real vector space and let us consider a structure of the Lie algebra defined by the brackets $[E_i, E_j] = C_{ij}^k E_k$, where $\{E_1, E_2, E_3, E_4\}$ is a basis of V and $C_{ij}^k \in \mathbb{R}$. Then, the Jacobi identity for C_{ij}^k

$$C_{ij}^k C_{ks}^l + C_{js}^k C_{ki}^l + C_{si}^k C_{kj}^l = 0 \qquad (2.1)$$

holds.

Let G be the associated real connected Lie group and let $\{X_1, X_2, X_3, X_4\}$ be a global basis of left-invariant vector fields induced by the basis of V.

We define an invariant almost complex structure on G by the conditions:

$$JX_1 = X_3, \quad JX_2 = X_4, \quad JX_3 = -X_1, \quad JX_4 = -X_2. \qquad (2.2)$$

Further, let us consider the left-invariant metric defined by

$$\begin{aligned} &g(X_1, X_1) = g(X_2, X_2) = -g(X_3, X_3) = -g(X_4, X_4) = 1, \\ &g(X_i, X_j) = 0 \text{ for } i \neq j. \end{aligned} \qquad (2.3)$$

The introduced metric is a Norden metric because of (2.2). In this way, the induced 4-dimensional manifold (G, J, g) is an almost complex manifold with Norden metric.

It is known [12] that a Lie group G, equipped with a complex structure J, is a complex Lie group if both, left and right translations on G are holomorphic maps. On the corresponding Lie algebra $\mathfrak{g}$ of G this condition is equivalent to

$$[X, JY] = J[X, Y] \quad \text{for all } X, Y \in \mathfrak{g}, \tag{2.4}$$

i.e. $\mathfrak{g}$ is a complex Lie algebra. In this case the complex structure J is called *bi-invariant* [6].

Let J, defined by (2.2), be a bi-invariant complex structure. Then, obviously, (G, J, g) is a complex manifold with Norden metric. By (2.2) and (2.4) we obtain the following conditions for the commutators of the basic vector fields

$$\begin{aligned} &[X_1, X_4] = -[X_2, X_3] = J[X_1, X_2] = -J[X_3, X_4], \\ &[X_1, X_3] = [X_2, X_4] = 0. \end{aligned} \tag{2.5}$$

Thus, we can put the non-zero Lie brackets

$$\begin{aligned} &[X_1, X_2] = -[X_3, X_4] = \lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3 + \lambda_4 X_4, \\ &[X_2, X_3] = -[X_1, X_4] = \lambda_3 X_1 + \lambda_4 X_2 - \lambda_1 X_3 - \lambda_2 X_4, \end{aligned} \tag{2.6}$$

where λ_i $(i = 1, 2, 3, 4)$ are real parameters. By direct computations we prove that the commutators (2.6) satisfy the Jacobi identity. Therefore, the conditions (2.6) define a 4-parametric family of 4-dimensional real Lie algebras $\mathfrak{g}$.

Further, let us recall that a Lie algebra $\mathfrak{g}$ is said to be *solvable* if its derived series

$$\mathfrak{D}^0\mathfrak{g} = \mathfrak{g},\ \mathfrak{D}^1\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}], \dots,\ \mathfrak{D}^{k+1}\mathfrak{g} = [\mathfrak{D}^k\mathfrak{g}, \mathfrak{D}^k\mathfrak{g}], \dots$$

vanishes for some $k \in \mathbb{N}$. Then, having in mind (2.6), it is easy to check that $\mathfrak{D}^2\mathfrak{g} = \{0\}$ and thus the Lie algebras $\mathfrak{g}$ are solvable.

2.1 Geometric characteristics of the constructed manifold

First, we establish the following

Theorem 2.1 *Let (G, J, g) be the 4-dimensional complex manifold with Norden metric and bi-invariant complex structure constructed by (2.2), (2.3) and (2.4), and let $\mathfrak{g}$ be the associated Lie algebra of G introduced by (2.6). Then, (G, J, g) is a Kähler manifold with Norden metric.*

Proof. Let ∇ be the Levi-Civita connection of g. Then, the following well-known condition is valid

$$\begin{aligned} 2g(\nabla_X Y, Z) = &Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) \\ &+g([X, Y], Z) + g([Z, X], Y) + g([Z, Y], X). \end{aligned} \tag{2.7}$$

Applying (2.2), (2.3) and the fact that J is a bi-invariant complex structure to (2.7), we obtain

$$
\begin{aligned}
&2g\left((\nabla_{X_i}J)X_j, X_k\right) = g\left([X_i, JX_j] - J[X_i, X_j], X_k\right)\\
&+g\left([X_k, JX_i] - [JX_k, X_i], X_j\right) + g\left(J[X_k, X_j] - [JX_k, X_j], X_i\right) = 0
\end{aligned}
$$

for all $i, j, k = 1, 2, 3, 4$, i.e. $\nabla J = 0$ on $\mathfrak{g}$. Therefore, by (1.8), the manifold (G, J, g) belongs to the class $\mathcal{W}_0$ of the Kähler manifolds with Norden metric. ■

Next, by (2.3), (2.6) and (2.7) we find the components of the Levi-Civita connection on the considered manifold as follows:

$$
\begin{aligned}
&\nabla_{X_1}X_1 = -\nabla_{X_3}X_3 = -\lambda_1 X_2 - \lambda_3 X_4,\\
&\nabla_{X_1}X_2 = -\nabla_{X_3}X_4 = \lambda_1 X_1 + \lambda_3 X_3,\\
&\nabla_{X_1}X_3 = \nabla_{X_3}X_1 = \lambda_3 X_2 - \lambda_1 X_4,\\
&\nabla_{X_1}X_4 = \nabla_{X_3}X_2 = -\lambda_3 X_1 + \lambda_1 X_3,\\
&\nabla_{X_2}X_1 = -\nabla_{X_4}X_3 = -\lambda_2 X_2 - \lambda_4 X_4,\\
&\nabla_{X_2}X_2 = -\nabla_{X_4}X_4 = \lambda_2 X_1 + \lambda_4 X_3,\\
&\nabla_{X_2}X_3 = \nabla_{X_4}X_1 = \lambda_4 X_2 - \lambda_2 X_4,\\
&\nabla_{X_2}X_4 = \nabla_{X_4}X_2 = -\lambda_4 X_1 + \lambda_2 X_3.
\end{aligned}
\tag{2.8}
$$

2.2 Curvature properties of the constructed manifold

Let R be the curvature tensor on (G, J, g) determined by (1.10). The condition $\nabla J = 0$ implies $R(X, Y)JZ = JR(X, Y)Z$ and thus $R(X, Y, JZ, JW) = -R(X, Y, Z, W)$ for any $X, Y, Z, W \in \mathfrak{g}$, i.e. R is a Kähler tensor. We denote its components by $R_{ijks} = R(X_i, X_j, X_k, X_s)$ $(i, j, k, s = 1, 2, 3, 4)$. Then, using (2.8), we get the following non-zero components of R:

$$
\begin{aligned}
&R_{1441} = R_{2332} = R_{1423} = -R_{1221} = -R_{3443} = -R_{1234} = \lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2,\\
&R_{1241} = R_{2132} = -R_{3243} = -R_{4134} = 2\left(\lambda_1\lambda_3 + \lambda_2\lambda_4\right).
\end{aligned}
\tag{2.9}
$$

Further, taking into account that the inverse matrix of g has the form

$$
\left(g^{ij}\right) = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{2.10}
$$

we obtain by (1.11) and (2.9) the non-zero components $\rho_{ij} = \rho(X_i, X_j)$ of the

Ricci tensor and the values of the scalar curvatures τ and $\overset{*}{\tau}$ as follows:

$$\begin{aligned}
&\rho_{11} = \rho_{22} = -\rho_{33} = -\rho_{44} = -2\left(\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2\right),\\
&\rho_{13} = \rho_{24} = 4\left(\lambda_1\lambda_3 + \lambda_2\lambda_4\right),\\
&\tau = -8\left(\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2\right), \quad \overset{*}{\tau} = 16\left(\lambda_1\lambda_3 + \lambda_2\lambda_4\right).
\end{aligned} \tag{2.11}$$

Obviously, the scalar curvatures of (G, J, g) are constant.

Let us consider the characteristic 2-planes α_{ij} spanned by the basic vectors $\{X_i, X_j\}$ at an arbitrary point of the manifold:

- totally real 2-planes - α_{12}, α_{14}, α_{23}, α_{34};
- holomorphic 2-planes - α_{13}, α_{24}.

Then, having in mind (1.15), (1.17) and (2.9), we obtain the corresponding sectional curvatures:

$$\begin{aligned}
&\nu(\alpha_{12}) = \nu(\alpha_{34}) = \nu(\alpha_{14}) = \nu(\alpha_{23}) = -\left(\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2\right),\\
&\overset{*}{\nu}(\alpha_{12}) = \overset{*}{\nu}(\alpha_{34}) = \overset{*}{\nu}(\alpha_{14}) = \overset{*}{\nu}(\alpha_{23}) = 2\left(\lambda_1\lambda_3 + \lambda_2\lambda_4\right),\\
&\nu(\alpha_{13}) = \nu(\alpha_{24}) = \overset{*}{\nu}(\alpha_{13}) = \overset{*}{\nu}(\alpha_{24}) = 0.
\end{aligned} \tag{2.12}$$

Thus, (G, J, g) is of constant totally real sectional curvatures.

By (1.18) and (2.9) we establish that the holomorphic bisectional curvature of the unique pair of basic holomorphic 2-planes $\{\alpha_{13}, \alpha_{24}\}$ vanishes, i.e. $h(X_1, X_2) = 0$.

It has been proved [1] that a Kähler manifold with Norden metric (M, J, g) ($\dim M = 2n \geq 4$) is of pointwise constant sectional curvatures ν and $\overset{*}{\nu}$ for any non-degenerate totally real 2-plane α in T_pM, if and only if

$$R = \nu\left\{\pi_1 - \pi_2\right\} + \overset{*}{\nu}\pi_3. \tag{2.13}$$

Both functions ν and $\overset{*}{\nu}$ are constant if M is connected and $\dim M \geq 6$.

Then, by this statement and (2.12) we obtain

Theorem 2.2 *The curvature tensor R of the Kähler manifold (G, J, g) has the form (2.13).*

It is clear that the equations (2.12) and (2.13) immediately imply $\nabla R = 0$, i.e. the manifold (G, J, g) is locally symmetric.

The form (2.13) of R implies

$$\rho = \frac{1}{4}\left\{\tau g - \overset{*}{\tau}\widetilde{g}\right\}, \qquad \nu = \frac{\tau}{8}, \quad \overset{*}{\nu} = \frac{\overset{*}{\tau}}{8}. \tag{2.14}$$

Then, taking into account (1.15), (1.16), (2.13) and (2.14), we obtain the Weyl tensor of the manifold as follows

$$W = \frac{\tau}{24}\left\{\pi_1 - 3\pi_2\right\}. \tag{2.15}$$

Theorem 2.2 and the equalities (2.11), (2.14) and (2.15) imply

Corollary 2.3 *The following conditions are equivalent*:

(i) $R = \frac{\tau}{8}\{\pi_1 - \pi_2\}$;

(ii) $\overset{*}{\tau} = 0$;

(iii) $\lambda_3 = k\lambda_2,\ \lambda_4 = -k\lambda_1,\ |k| \neq 1$ *or* $\lambda_2 = k\lambda_1,\ \lambda_3 = -k\lambda_4, |\lambda_1| \neq |\lambda_4|$;

(iv) $\rho = \frac{\tau}{4} g$, *i.e. the manifold is Einsteinian.*

Corollary 2.4 *The following conditions are equivalent*:

(i) $R = \frac{\overset{*}{\tau}}{8}\pi_3$;

(ii) $\tau = 0$;

(iii) $|\lambda_1| = |\lambda_3|,\ |\lambda_2| = |\lambda_4|$ *or* $\lambda_1 = \lambda_4,\ \lambda_2 = \lambda_3$ *or* $\lambda_1 = -\lambda_4,\ \lambda_2 = -\lambda_3$;

(iv) *the Weyl tensor vanishes.*

It is clear that the manifold (G, J, g) in Corollary 2.4 is conformally equivalent to a flat manifold.

3 A Lie group as a 4-dimensional complex manifold with Norden metric

Let G be a real connected Lie group, and let $\mathfrak{g}$ be its Lie algebra. If $\{X_1, X_2, X_3, X_4\}$ is a global basis of left-invariant vector fields of G, we define an invariant almost complex structure J and a left-invariant Norden metric g on G by the conditions (2.2) and (2.3), respectively. Then, as in the previous section, (G, J, g) is an almost complex manifold with Norden metric.

It is known [3] that an almost complex structure J on a Lie group G is said to be *abelian* if

$$[JX, JY] = [X, Y] \quad \text{for all } X, Y \in \mathfrak{g}. \tag{3.1}$$

It follows from (3.1) that the Nijenhuis tensor vanishes on $\mathfrak{g}$, i.e. J is a complex structure. Thus, (G, J, g) is a complex manifold with Norden metric, i.e. $(G, J, g) \in \mathcal{W}_1 \oplus \mathcal{W}_2$.

Now, let us consider the Lie algebra $\mathfrak{g}$ of G. If J, determined by (2.2), is an abelian complex structure we obtain

Proposition 3.1 *Let (G, J, g) be a 4-dimensional complex manifold with Norden metric and abelian complex structure defined by (2.2) and (2.3). Then the Lie algebra $\mathfrak{g}$ of G is given as follows:*

$$\begin{aligned} &[X_1, X_2] = [X_3, X_4], \ \textit{i.e.}\ C_{12}^k = C_{34}^k, \\ &[X_1, X_4] = [X_2, X_3], \ \textit{i.e.}\ C_{14}^k = C_{23}^k, \\ &[X_1, X_3] = C_{13}^k X_k, \qquad [X_2, X_4] = C_{24}^k X_k, \end{aligned} \tag{3.2}$$

where $C_{ij}^k \in \mathbb{R}$ $(i, j, k = 1, 2, 3, 4)$ satisfy the Jacobi identity.

Further, we construct our example by putting

$$C_{12}^k = C_{34}^k = C_{14}^k = C_{23}^k = 0, \quad k = 1, 2, 3, 4.$$

In this case the Jacobi identity (2.1) implies

$$\begin{aligned} [X_1, X_3] &= \lambda_1 X_1 + \lambda_3 X_3, \\ [X_2, X_4] &= \lambda_2 X_2 + \lambda_4 X_4, \end{aligned} \tag{3.3}$$

where $\lambda_i \in \mathbb{R}$ $(i = 1, 2, 3, 4)$. Thus, the conditions (3.3) define a family of 4-dimensional real Lie algebras $\mathfrak{g}$, which is characterized by four parameters.

It has been proved [3] that if a Lie algebra $\mathfrak{g}$ admits an abelian complex structure then $\mathfrak{g}$ is solvable. Therefore, the above considered Lie algebras (3.3) are solvable.

3.1 Geometric characteristics of the constructed manifold

Having in mind (2.3), (2.7) and (3.3) we obtain the following non-zero components of the Levi-Civita connection of (G, J, g):

$$\begin{aligned} \nabla_{X_1} X_1 &= \lambda_1 X_3, & \nabla_{X_2} X_2 &= \lambda_2 X_4, \\ \nabla_{X_1} X_3 &= \lambda_1 X_1, & \nabla_{X_2} X_4 &= \lambda_2 X_2, \\ \nabla_{X_3} X_1 &= -\lambda_3 X_3, & \nabla_{X_4} X_2 &= -\lambda_4 X_4, \\ \nabla_{X_3} X_3 &= -\lambda_3 X_1, & \nabla_{X_4} X_4 &= -\lambda_4 X_2. \end{aligned} \tag{3.4}$$

Then, by (2.2) and (3.4) we obtain the following non-zero components of ∇J:

$$\begin{aligned} (\nabla_{X_1} J) X_1 &= 2\lambda_1 X_1, & (\nabla_{X_2} J) X_2 &= 2\lambda_2 X_2, \\ (\nabla_{X_1} J) X_3 &= -2\lambda_1 X_3, & (\nabla_{X_2} J) X_4 &= -2\lambda_2 X_4, \\ (\nabla_{X_3} J) X_1 &= -2\lambda_3 X_1, & (\nabla_{X_4} J) X_2 &= -2\lambda_4 X_2, \\ (\nabla_{X_3} J) X_3 &= 2\lambda_3 X_3, & (\nabla_{X_4} J) X_4 &= 2\lambda_4 X_4. \end{aligned} \tag{3.5}$$

Next, taking into account (1.2), (2.3) and (3.5), we get the non-zero components $F_{ijk} = F(X_i, X_j, X_k)$ of F as follows:

$$\begin{aligned} F_{111} &= F_{133} = 2\lambda_1, & F_{222} &= F_{244} = 2\lambda_2, \\ F_{311} &= F_{333} = -2\lambda_3, & F_{422} &= F_{444} = -2\lambda_4. \end{aligned} \tag{3.6}$$

According to (1.4), (2.10) and (3.6), the components $\theta_i = \theta(X_i)$ and $\theta_i^* = \theta^*(X_i)$ of the Lie forms θ and $\theta^* = \theta \circ J$ are the following:

$$\begin{aligned} &\theta_1 = 2\lambda_1, \quad &\theta_2 = 2\lambda_2, \quad &\theta_3 = 2\lambda_3, \quad &\theta_4 = 2\lambda_4; \\ &\theta_1^* = 2\lambda_3, \quad &\theta_2^* = 2\lambda_4, \quad &\theta_3^* = -2\lambda_1, \quad &\theta_4^* = -2\lambda_2. \end{aligned} \tag{3.7}$$

By (3.4) and (3.7) we get that $(\nabla_{X_i}\theta^*)X_j = (\nabla_{X_j}\theta^*)X_i$, $(i,j=1,2,3,4)$, i.e. the Lie form θ^* is closed on (G,J,g).

Let Ω be the corresponding Lie vector of θ. Then, $J\Omega$ corresponds to θ^* as its Lie vector and by (1.5), (2.3), (2.10), (3.4) and (3.7) we have

$$\theta(\Omega) = 2\mathrm{div}(J\Omega) = 4\left(\lambda_1^2+\lambda_2^2-\lambda_3^2-\lambda_4^2\right), \tag{3.8}$$

where $\mathrm{div}(J\Omega) = g^{ij}(\nabla_{X_i}\theta^*)X_j$.

Next, in view of (1.19), we obtain from (2.3), (2.10) and (3.5) the square norm of ∇J as

$$\|\nabla J\|^2 = 8\left(\lambda_1^2+\lambda_2^2-\lambda_3^2-\lambda_4^2\right). \tag{3.9}$$

Having in mind Definition 1.3, the last equality implies

Proposition 3.2 *The manifold (G,J,g) is isotropic Kählerian if and only if the condition $\lambda_1^2+\lambda_2^2-\lambda_3^2-\lambda_4^2=0$ holds.*

3.2 Curvature properties of the constructed manifold

Let R be the curvature tensor of type $(0,4)$ of (G,J,g). By (1.10) and (3.4) we get the non-zero components of R as follows:

$$R_{1331} = -\left(\lambda_1^2-\lambda_3^2\right), \qquad R_{2442} = -\left(\lambda_2^2-\lambda_4^2\right). \tag{3.10}$$

Then, according to (2.2), (3.10) and Definition 1.4, we obtain

Theorem 3.3 *The curvature tensor R of (G,J,g) belongs to the class $\mathfrak{L}_1'$ and has the form*

$$R(X,Y,Z,W) = -\frac{1}{4}g\left((\nabla_X J)Y-(\nabla_Y J)X,(\nabla_Z J)W-(\nabla_W J)Z\right). \tag{3.11}$$

Proof. Let $X=x^iX_i$, $Y=y^iX_i$, $Z=z^iX_i$, $W=w^iX_i$, where $x^i,y^i,z^i,w^i\in\mathbb{R}$ $(i=1,2,3,4)$, be arbitrary vectors in $\mathfrak{g}$. By (3.10) we have

$$\begin{aligned} R(X,Y,Z,W) = &\left(\lambda_1^2-\lambda_3^2\right)\left(x^1y^3-x^3y^1\right)\left(z^1w^3-z^3w^1\right)\\ &+\left(\lambda_2^2-\lambda_4^2\right)\left(x^2y^4-x^4y^2\right)\left(z^2w^4-z^4w^2\right). \end{aligned} \tag{3.12}$$

Then, (1.28) and (3.5) imply that the right-hand side of (3.11) is equal to that of (3.12). ■

Proposition 3.4 *The curvature tensor R of the manifold (G,J,g) satisfies*

$$R(X,Y,Z,W) = g\left([X,Y],[Z,W]\right). \tag{3.13}$$

Proof. The validity of (3.13) follows from (3.3) and (3.12) by direct computations as in Theorem 3.3. ■

Further, according to (3.4) and (3.10), we obtain

$$\big(\nabla_{X_i} R\big)(X_j, X_k, X_l, X_s) = 0 \tag{3.14}$$

for all $i, j, k, l, s = 1, 2, 3, 4$. Thus, we establish

Proposition 3.5 *The manifold (G, J, g) is locally symmetric.*

By (1.11), (2.10) and (3.10) we compute the non-zero components $\rho_{ij} = \rho(X_i, X_j)$ of the Ricci tensor and the value of the scalar curvature τ:

$$\rho_{11} = -\rho_{33} = \lambda_1^2 - \lambda_3^2, \quad \rho_{22} = -\rho_{44} = \lambda_2^2 - \lambda_4^2, \quad \tau = 2\left(\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2\right). \tag{3.15}$$

Proposition 3.2 and (3.15) imply

Proposition 3.6 *The manifold (G, J, g) is isotropic Kählerian if and only if it is scalar flat.*

Taking into account (2.3) and (3.15), we prove the following

Theorem 3.7 *The manifold (G, J, g) is Einstein if and only if the conditions $|\lambda_1| = |\lambda_2|$, $|\lambda_3| = |\lambda_4|$ hold.*

Let α_{ij} be a non-degenerate 2-plane spanned by the basic vectors $\{X_i, X_j\}$ at an arbitrary point of the manifold. Then, by (1.15), (1.17) and (3.10), we obtain the corresponding sectional curvatures as follows:

$$\begin{aligned} &\nu(\alpha_{13}) = \lambda_1^2 - \lambda_3^2, \qquad \nu(\alpha_{24}) = \lambda_2^2 - \lambda_4^2, \\ &\nu(\alpha_{12}) = \nu(\alpha_{14}) = \nu(\alpha_{23}) = \nu(\alpha_{34}) = 0, \end{aligned} \tag{3.16}$$

i.e. (G, J, g) is of vanishing totally real sectional curvatures. Moreover, (3.16) and Theorem 3.7 imply

Proposition 3.8 *The manifold (G, J, g) is of constant non-zero holomorphic sectional curvatures if and only if it is Einstein.*

Taking into account (1.15), (1.18) and (3.10), we establish that the holomorphic bisectional curvature of the unique pair of basic holomorphic 2-planes $\{\alpha_{13}, \alpha_{24}\}$ vanishes.

Now, let us consider the Weyl tensor W on (G, J, g). From (1.15), (3.10) and (3.15), we get the non-zero components of W as follows:

$$\begin{aligned} W_{1331} &= W_{2442} = 2W_{1221} = 2W_{3443} = -2W_{1441} \\ &= -2W_{2332} = -\tfrac{1}{3}\left(\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2\right). \end{aligned} \tag{3.17}$$

Hence, (1.16), Proposition 3.2 and Proposition 3.6 imply

Proposition 3.9 *The manifold (G, J, g) is isotropic Kählerian if and only if the Weyl tensor vanishes. In this case the curvature tensor has the form*

$$R = \frac{1}{2}\psi_1(\rho). \tag{3.18}$$

Having in mind (3.8) and Prepositions 3.2, 3.6 and 3.9, we finally obtain

Theorem 3.10 *The following conditions are equivalent*:

(i) (G, J, g) *is isotropic Kählerian*;

(ii) (G, J, g) *is scalar flat*;

(iii) *the condition* $\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2 = 0$ *holds*;

(iv) *the curvature tensor has the form* $R = \frac{1}{2}\psi_1(\rho)$;

(v) *the Weyl tensor vanishes*;

(vi) $\theta(\Omega) = 2\mathrm{div}(J\Omega) = 0$.

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