On a class complex manifolds with Norden metric

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Abstract

A subclass of one of the basic classes complex manifolds with Norden metric is introduced. Some curvature properties of 4-dimensional manifolds belonging to this subclass are studied. A condition this manifolds to be conformally flat is given. An example of such 4-dimensional manifold is constructed on a Lie group. The manifold obtained in this way is proved to be Einstein.

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1 Preliminaries

Let (M, J, g) be a $2n$-dimensional almost complex manifold with Norden metric, i.e. J is an almost complex structure and g is a metric on M such that:

$$J^2X = -X, \qquad g(JX, JY) = -g(X, Y), \qquad X, Y \in \mathfrak{X}(M). \tag{1.1}$$

The associated metric $\widetilde{g}$ of g on M, given by $\widetilde{g}(X, Y) = g(X, JY)$, is a Norden metric, too. Both metrics are necessarily of signature (n, n).

Further, X, Y, Z, W (x, y, z, w, respectively) will stand for arbitrary differentiable vector fields on M (vectors in T_pM, $p \in M$, respectively).

Let ∇ be the Levi-Civita connection of the metric g. Then, the tensor field F of type $(0, 3)$ on M is defined by $F(X, Y, Z) = g\left((\nabla_X J)Y, Z\right)$ and has the following symmetries

$$F(X, Y, Z) = F(X, Z, Y) = F(X, JY, JZ). \tag{1.2}$$

Let $\{e_i\}$ $(i = 1, 2, \ldots, 2n)$ be an arbitrary basis of T_pM at a point p of M. The components of the inverse matrix of g are denoted by g^{ij} with respect to the basis $\{e_i\}$.

The Lie forms θ and θ^* associated with F are defined by

$$\theta(z) = g^{ij}F(e_i, e_j, z), \qquad \theta^* = \theta \circ J, \tag{1.3}$$

and the corresponding Lie vector is denoted by Ω, i.e. $\theta(z) = g(z, \Omega)$.

A classification of the almost complex manifolds with Norden metric is introduced in [2], where eight classes of these manifolds are characterized according to the properties of F. The three basic classes are given as follows:

$$\begin{aligned}
&\mathcal{W}_1 : F(X, Y, Z) = \tfrac{1}{2n}\left[g(X, Y)\theta(Z) + g(X, Z)\theta(Y)\right.\\
&\qquad\qquad\qquad \left.+g(X, JY)\theta(JZ) + g(X, JZ)\theta(JY)\right];\\
&\mathcal{W}_2 : F(X, Y, JZ) + F(Y, Z, JX) + F(Z, X, JY) = 0, \quad \theta = 0;\\
&\mathcal{W}_3 : F(X, Y, Z) + F(Y, Z, X) + F(Z, X, Y) = 0;
\end{aligned} \tag{1.4}$$

Let R be the curvature tensor of ∇, i.e. $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$ and $R(X,Y,Z,W) = g\left(R(X,Y)Z,W\right)$.

A tensor L of type $(0,4)$ is called *curvature-like* if it satisfies the following conditions for any $X,Y,Z,W \in \mathfrak{X}(M)$: $L(X,Y,Z,W) = -L(Y,X,Z,W) = -L(X,Y,W,Z)$, $L(X,Y,Z,W) + L(Y,Z,X,W) + L(Z,X,Y,W) = 0$.

The Ricci tensor $\rho(L)$ and the scalar curvatures $\tau(L)$ and $\tau^*(L)$ of L are defined by:

$$\rho(L)(y,z) = g^{ij} L(e_i, y, z, e_j),\ \ \tau(L) = g^{ij}\rho(L)(e_i,e_j),\ \ \tau^*(L) = g^{ij}\rho(L)(e_i, Je_j). \tag{1.5}$$

A curvature-like tensor L is said to be *a Kählerian* if $L(X,Y,JZ,JW) = -L(X,Y,Z,W)$.

Let S be a symmetric tensor of type $(0,2)$. We consider the following curvature-like tensors of type $(0,4)$:

$$\begin{aligned} \psi_1(S)(X,Y,Z,W) &= g(Y,Z)S(X,W) - g(X,Z)S(Y,W) \\ &\quad + g(X,W)S(Y,Z) - g(Y,W)S(X,Z); \qquad \pi_1 = \tfrac{1}{2}\psi_1(g); \\ \pi_2(X,Y,Z,W) &= g(Y,JZ)g(X,JW) - g(X,JZ)g(Y,JW); \qquad \pi_3 = -\psi_1\left(\widetilde{g}\right). \end{aligned} \tag{1.6}$$

The Weyl tensor W of R is defined as usually by

$$W(R) = R - \frac{1}{2n-2}\left\{\psi_1(\rho) - \frac{\tau}{2n-1}\pi_1\right\}. \tag{1.7}$$

It is known that the Weyl tensor vanishes if and only if the manifold is conformally flat.

Let $\alpha = \{x,y\}$ be a non-degenerate 2-plane spanned by the vectors $x, y \in T_pM$, $p \in M$. The sectional curvatures of α with respect to the curvature-like tensor L are given by

$$\nu(L;p) = \frac{L(x,y,y,x)}{\pi_1(x,y,y,x)}, \qquad \nu^*(L;p) = \frac{L(x,y,y,x)}{\pi_1(x,y,y,x)}. \tag{1.8}$$

Let us note that the square norm of ∇J is defined by

$$\|\nabla J\|^2 = g^{ij} g^{kl} g\left((\nabla_{e_i} J)e_k, (\nabla_{e_j} J)e_l\right). \tag{1.9}$$

Definition 1.1. [3] An almost complex manifold with Norden metric satisfying the condition $\|\nabla J\|^2 = 0$ is said to be an *isotropic Kähler manifold with Norden metric.*

2 Curvature properties of 4-dimensional $\mathcal{W}_1^*$-manifolds

Let (M,J,g) be a $\mathcal{W}_1$-manifold with closed the Lie form θ^*, i.e. $\left(\nabla_X \theta\right) JY = \left(\nabla_Y \theta\right) JX$. We denote the class of these manifolds by $W_1^* \subset \mathcal{W}_1$.

In [6] is introduced the tensor R^* by

$$R^* = R - \frac{1}{2n}\psi_1(S), \qquad S(X,Y) = \big(\nabla_X \theta\big)JY + \frac{1}{2n}\theta(X)\theta(Y) + \frac{\theta(\Omega)}{4n}g(X,Y). \tag{2.1}$$

By the fact that S is symmetric on a $\mathcal{W}_1^*$-manifold we conclude that R^* is a curvature-like tensor. It is proved [6] that in this case the tensor R^* is Kählerian and $W(R) = W(R^*)$.

In [7] is established the following

Theorem 2.1. [7] *Let (M, J, g) be a 4-dimensional almost complex manifold with Norden metric and let L be a Kähler tensor on M. Then L has the following form*

$$L = \nu(L)\{\pi_1 - \pi_2\} + \nu^*(L)\pi_3, \qquad \nu(L) = \frac{\tau(L)}{8}, \quad \nu^*(L) = \frac{\tau^*(L)}{8}. \tag{2.2}$$

Then, by (1.7), (2.1) and Theorem 2.1 we obtain

Theorem 2.2. *Let (M, J, g) be a 4-dimensional $\mathcal{W}_1^*$-manifold. Then, the curvature tensor R and the Weyl tensor $W(R)$ have the forms, respectively*

$$R = \frac{\tau_*}{8}\{\pi_1 - \pi_2\} - \frac{\tau_*}{12}\pi_1 + \frac{1}{2}\big\{\psi_1(\rho) - \frac{\tau}{3}\pi_1\big\}, \quad W(R) = \frac{\tau_*}{24}\{\pi_1 - 3\pi_2\}, \tag{2.3}$$

where $\tau_* = \tau(R^*) = \tau - \frac{3}{2}\big[\mathrm{div}(J\Omega) - \frac{\theta(\Omega)}{4}\big]$ *and* $\mathrm{div}(J\Omega) = \nabla_i J_k^i \Omega^k$.

By Theorem 2.2 we obtain

Theorem 2.3. *The Weyl tensor of a 4-dimensional $\mathcal{W}_1^*$-manifold vanishes if and only if the condition $\tau = \frac{3}{2}\big[\mathrm{div}(J\Omega) - \frac{\theta(\Omega)}{4}\big]$ holds.*

It is known that a 4-dimensional almost complex manifold with Norden metric is called *a space form* if its curvature tensor has the form $R = \frac{\tau}{12}\pi_1$. Obviously, such manifolds are Einstein, locally symmetric and conformally flat.

Corollary 2.4. *If a 4-dimensional $\mathcal{W}_1^*$-manifold is a space form, $\tau = \frac{3}{2}\big[\mathrm{div}(J\Omega) - \frac{\theta(\Omega)}{4}\big]$.*

It has been proved [7] that on a $\mathcal{W}_1$-manifold it is valid

$$\|\nabla J\|^2 = \frac{2}{n}\theta(\Omega). \tag{2.4}$$

Then, by Theorem 2.2, Definition 1.1 and (2.4) it follows immediately

Corollary 2.5. *Let (M, J, g) be a 4-dimensional isotropic Kähler $\mathcal{W}_1^*$-manifold. Then, its curvature tensor has the form $R = \frac{2\tau - 3\mathrm{div}(J\Omega)}{48}\{\pi_1 - 3\pi_2\} + \frac{1}{2}\big\{\psi_1(\rho) - \frac{\tau}{3}\pi_1\big\}$.*

3 A Lie group as a 4-dimensional $\mathcal{W}_1^*$-manifold

Let $\mathfrak{g}$ be a real 4-dimensional Lie algebra corresponding to a real connected Lie group G. If $\{X_1, X_2, X_3, X_4\}$ is a global basis of left invariant vector fields on G and $[X_i, X_j] = C_{ij}^k X_k$, then the Jacobi identity for C_{ij}^k holds:

$$C_{ij}^k C_{ks}^l + C_{js}^k C_{ki}^l + C_{si}^k C_{kj}^l = 0. \tag{3.1}$$

We define an almost complex structure on G by the conditions:

$$JX_1 = X_3, \quad JX_2 = X_4, \quad JX_3 = -X_1, \quad JX_4 = -X_2. \tag{3.2}$$

Let us consider the left-invariant metric given by

$$
\begin{aligned}
&g(X_1,X_1)=g(X_2,X_2)=-g(X_3,X_3)=-g(X_4,X_4)=1,\\
&g(X_i,X_j)=0 \text{ for } i\neq j.
\end{aligned}
\tag{3.3}
$$

The introduced metric Norden because of (3.2). In this way, the induced 4-dimensional manifold (G,J,g) is an almost complex manifold with Norden metric.

Further, from the well-known equality

$$
\begin{aligned}
2g(\nabla_X Y,Z)&=Xg(Y,Z)+Yg(X,Z)-Zg(X,Y)\\
&\quad+g([X,Y],Z)+g([Z,X],Y)+g([Z,Y],X)
\end{aligned}
\tag{3.4}
$$

we obtain

$$
\begin{aligned}
&2F(X_i,X_j,X_k)=g\left([X_i,JX_j]-J[X_i,X_j],X_k\right)\\
&+g\left([X_k,JX_i]-[JX_k,X_i],X_j\right)+g\left(J[X_k,X_j]-[JX_k,X_j],X_i\right).
\end{aligned}
\tag{3.5}
$$

Let (G,J,g) be a $\mathcal{W}_1$-manifold. Then, by (1.2), (1.3), (1.4), (3.5) we prove

Proposition 3.1. *If (G,J,g) is a 4-dimensional $\mathcal{W}_1$-manifold, the Lie algebra $\mathfrak{g}$ of G is given by the conditions:*

$$
\begin{aligned}
&C_{13}^1=C_{23}^2-C_{12}^4=C_{14}^2-C_{34}^4, && C_{24}^1=-(C_{12}^4+C_{14}^2)=-(C_{23}^2+C_{34}^4),\\
&C_{13}^2=C_{12}^3-C_{23}^1=C_{34}^3-C_{14}^1, && C_{24}^2=C_{12}^3+C_{14}^1=C_{23}^1+C_{34}^3,\\
&C_{13}^3=C_{12}^2+C_{23}^4=C_{14}^4+C_{34}^2, && C_{24}^3=C_{34}^2-C_{23}^4=C_{12}^2-C_{14}^1,\\
&C_{13}^4=-(C_{14}^3+C_{34}^1)=-(C_{12}^1+C_{23}^2), && C_{24}^4=C_{14}^3-C_{12}^1=C_{23}^3-C_{34}^1,
\end{aligned}
\tag{3.6}
$$

where $C_{ij}^k\in\mathbb{R}$ $(i,j,k=1,2,3,4)$ satisfy the Jacobi identity (3.1).

One solution to (3.1) and (3.6) is the 4-parametric family of Lie algebras $\mathfrak{g}$ given by

$$
\begin{aligned}
&[X_1,X_2]=\lambda_1X_1+\lambda_2X_2, && [X_2,X_3]=\lambda_4X_2-\lambda_1X_3,\\
&[X_1,X_3]=\lambda_4X_1+\lambda_2X_3, && [X_2,X_4]=\lambda_3X_2-\lambda_1X_4,\\
&[X_1,X_4]=\lambda_3X_1+\lambda_2X_4, && [X_3,X_4]=\lambda_3X_3-\lambda_4X_4,
\end{aligned}
\tag{3.7}
$$

where $\lambda_i\in\mathbb{R}$ $(i=1,2,3,4)$. Thus, the equality (3.7) defines a 4-parametric family of 4-dimensional $\mathcal{W}_1$-manifolds.

If we put in (3.7) one of the parameters λ_i equal to one and the rest three equal to zero, we obtain the Lie algebra corresponding to the Lie group given as an example of a $\mathcal{W}_1$-manifold (in the case of dimension four) by R. Castro and L. M. Hervella [1].

It is well-known that a Lie algebra $\mathfrak{g}$ is *solvable* if its derived series

$$
\mathfrak{D}^0\mathfrak{g}=\mathfrak{g},\ \mathfrak{D}^1\mathfrak{g}=[\mathfrak{g},\mathfrak{g}],\ldots,\ \mathfrak{D}^{k+1}\mathfrak{g}=[\mathfrak{D}^k\mathfrak{g},\mathfrak{D}^k\mathfrak{g}],\ldots
$$

vanishes for some $k\in\mathbb{N}$. Then, having in mind (3.7), it is easy to check that $\mathfrak{D}^2\mathfrak{g}=\{0\}$ and thus the Lie algebras (3.7) are solvable.

Let us remark that the Killing form [4] $B(X,Y)=\mathrm{tr}(\mathrm{ad}X\mathrm{ad}Y)$ of the Lie algebras (3.7) is degenerate, i.e. $\det B=0$. Hence, it cannot be a Norden metric.

By (3.3), (3.4) and (3.7) we get the essential components of the Levi-Civita connection of the manifold (G,J,g) as follows:

$$
\begin{array}{lll}
\nabla_{X_1}X_1 = -\lambda_1X_2 + \lambda_4X_3 + \lambda_3X_4, & \nabla_{X_1}X_2 = \lambda_1X_1, & \nabla_{X_2}X_3 = \lambda_4X_2, \\
\nabla_{X_2}X_2 = \lambda_2X_1 + \lambda_4X_3 + \lambda_3X_4, & \nabla_{X_1}X_3 = \lambda_4X_1, & \nabla_{X_2}X_4 = \lambda_3X_2, \\
\nabla_{X_3}X_3 = -\lambda_2X_1 + \lambda_1X_2 - \lambda_3X_4, & \nabla_{X_1}X_4 = \lambda_3X_1, & \nabla_{X_3}X_4 = \lambda_3X_3. \\
\nabla_{X_4}X_4 = -\lambda_2X_1 + \lambda_1X_2 - \lambda_4X_3, & &
\end{array}
\tag{3.8}
$$

Next, by (3.2), (3.3) and (3.5) we compute the essential non-zero components $F_{ijk} = F(X_i,X_j,X_k)$ of the tensor F as follows:

$$
\begin{array}{ll}
-F_{114} = F_{312} = \frac{1}{2}F_{444} = \lambda_1, & F_{214} = F_{412} = \frac{1}{2}F_{333} = -\lambda_2, \\
F_{112} = F_{314} = \frac{1}{2}F_{222} = \lambda_3, & F_{212} = -F_{414} = \frac{1}{2}F_{111} = \lambda_4.
\end{array}
\tag{3.9}
$$

Having in mind (1.1), (1.3) and (3.9), we get the components $\theta_i = \theta(X_i)$ and $\theta_i^* = \theta^*(X_i)$ of the Lie forms θ and θ^*, respectively:

$$
\begin{array}{l}
\theta_1 = -\theta_3^* = 4\lambda_4, \quad \theta_2 = -\theta_4^* = 4\lambda_3, \quad \theta_3 = \theta_1^* = 4\lambda_2, \quad \theta_4 = \theta_2^* = -4\lambda_1, \\
\theta(\Omega) = \frac{4}{3}\mathrm{div}(J\Omega) = -16\big(\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2\big).
\end{array}
\tag{3.10}
$$

Then, by (3.8) and (3.10) we get $(\nabla_{X_i}\theta^*)X_j = (\nabla_{X_j}\theta^*)X_i$, $i,j = 1,2,3,4$. Hence the Lie form θ^* is closed and therefore we have

Proposition 3.2. *Let (G,J,g) be the 4-dimensional $\mathcal{W}_1$-manifold constructed by (3.2) and (3.3), and let $\mathfrak{g}$ be the Lie algebra of G defined by (3.7). Then $(G,J,g) \in \mathcal{W}_1^*$.*

By (2.4) and (3.10) we obtain the square norm of ∇J

$$
\|\nabla J\|^2 = -16\big(\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2\big). \tag{3.11}
$$

Then, having in mind Definition 1.1 and (3.11), we get

Proposition 3.3. *The manifold (G,J,g) is isotropic Kählerian if and only if the condition $\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2 = 0$ holds.*

By (3.3) and (3.8) we compute the non-zero components $R_{ijkl} = R(X_i,X_j,X_k,X_l)$ of the curvature tensor R as follows:

$$
-R_{1221} = R_{1331} = R_{1441} = R_{2332} = R_{2442} = -R_{3443} = \lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2. \tag{3.12}
$$

Let us consider the characteristic 2-planes α_{ij} spanned by the basic vectors $\{X_i,X_j\}$. By (1.6), (1.8), (3.3) and (3.12) we get the corresponding sectional curvatures as

$$
\nu(\alpha_{12}) = \nu(\alpha_{13}) = \nu(\alpha_{14}) = \nu(\alpha_{23}) = \nu(\alpha_{24}) = \nu(\alpha_{34}) = -\big(\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2\big). \tag{3.13}
$$

Then, according to the well-known Shur's Theorem [5], from (3.13) it follows

Proposition 3.4. *The curvature tensor of (G,J,g) has the form $R = \frac{\tau}{12}\pi_1$. Thus the manifold is Einstein.*

By (1.1), (1.5) and (3.10) we obtain the values of the scalar curvatures of the manifold $\tau = -12\big(\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2\big)$ and $\tau^* = 0$.

Finally, Proposition 3.3 and (3.12) give rise to

Proposition 3.5. *The manifold (G,J,g) is isotropic Kählerian if and only if it is flat.*

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