Curvature properties of conformal Kähler manifolds with Norden metric

Marta Kostadinova Teofilova

25.10.2005

Abstract

The class of the manifolds which are conformal equivalent to the Kähler manifolds with Norden metric is considered. The curvature tensor on such four-dimensional manifolds is obtained. The case of isotropic Kähler manifolds with Norden metric is studied. The transformation of the Levi-Civita connections of the both Norden metrics is considered. Some invariant tensors of this transformation are obtained.

Mathematics Subject Classification (2000): 53C15, 53C50.

1 Preliminaries

Let (M, J, g) be a $2n$-dimensional almost complex manifold with Norden metric, i.e. J is an almost complex structure and g is a metric on M such that:

$$J^2X = -X, \qquad g(JX, JY) = -g(X, Y), \qquad X, Y \in \mathfrak{X}(M). \tag{1.1}$$

The associated metric $\widetilde{g}$ of g on M given by $\widetilde{g}(X, Y) = g(X, JY)$ is a Norden metric, too. Both metrics are necessarily of signature (n, n).

Further, X, Y, Z, W (x, y, z, w, respectively) will stand for arbitrary differentiable vector fields on M (vectors in T_pM, $p \in M$, respectively).

Let ∇ be the Levi-Civita connection of the metric g. Then, the tensor field F of type $(0, 3)$ on M is defined by

$$F(X, Y, Z) = g\left((\nabla_X J)Y, Z\right). \tag{1.2}$$

It has the following symmetries

$$F(X, Y, Z) = F(X, Z, Y) = F(X, JY, JZ). \tag{1.3}$$

Let $\{e_i\}$ $(i = 1, 2, \dots, 2n)$ be an arbitrary basis of T_pM at a point p of M. The components of the inverse matrix of g are denoted by g^{ij} with respect to the basis $\{e_i\}$.

The Lie form θ associated with F is defined by

$$\theta(z) = g^{ij}F(e_i, e_j, z) \tag{1.4}$$

and the corresponding Lie vector is denoted by Ω, i.e. $\theta(z) = g(z, \Omega)$.

A classification of the considered manifolds with respect to the tensor F is given in [1]. Eight classes of almost complex manifolds with Norden metric are characterized there according to the properties of F. The three basic classes W_1, W_2, W_3 and the class $W_1 \oplus W_2$ of the complex manifolds with Norden metric are given as follows:

$$\begin{aligned} W_1 : F(X,Y,Z) = \tfrac{1}{2n}\, & [g(X,Y)\theta(Z) + g(X,Z)\theta(Y) \\ & + g(X,JY)\theta(JZ) + g(X,JZ)\theta(JY)]\,; \end{aligned} \tag{1.5}$$

$$W_2 : F(X,Y,JZ) + F(Y,Z,JX) + F(Z,X,JY) = 0, \quad \theta = 0;$$
$$W_3 : F(X,Y,Z) + F(Y,Z,X) + F(Z,X,Y) = 0;$$

$$W_1 \oplus W_2 : F(X,Y,JZ) + F(Y,Z,JX) + F(Z,X,JY) = 0. \tag{1.6}$$

The special class W_0 of the Kähler manifolds with Norden metric belonging to any other class is determined by the condition $F = 0$.

Let R be the curvature tensor of ∇, i.e.

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{1.7}$$

The corresponding tensor of type $(0,4)$ is denoted by the same letter and is given by $R(X,Y,Z,W) = g\,(R(X,Y)Z,W)$.

A tensor L of type $(0,4)$ is called *a curvature-like tensor* if it satisfies the following conditions for any $X,Y,Z,W \in \mathfrak{X}(M)$:

$$L(X,Y,Z,W) = -L(Y,X,Z,W) = -L(X,Y,W,Z),$$
$$L(X,Y,Z,W) + L(Y,Z,X,W) + L(Z,X,Y,W) = 0.$$

Then, the Ricci tensor $\rho(L)$ and the scalar curvatures $\tau(L)$ and $\tau^*(L)$ of L are defined by:

$$\rho(L)(y,z) = g^{ij} L(e_i,y,z,e_j); \quad \tau(L) = g^{ij}\rho(L)(e_i,e_j); \quad \tau^*(L) = g^{ij}\rho(L)(e_i,Je_j). \tag{1.8}$$

A curvature-like tensor L is called *a Kähler tensor* if it satisfies the condition

$$L(X,Y,JZ,JW) = -L(X,Y,Z,W), \qquad X,Y,Z,W \in \mathfrak{X}(M). \tag{1.9}$$

Further, let S be a symmetric and hybrid with respect to J tensor of type $(0,2)$, i.e. $S(JX,Y) = S(JY,X)$. We consider the following curvature-like tensors of type $(0,4)$:

$$\begin{aligned} \psi_1(S)(X,Y,Z,W) &= g(Y,Z)S(X,W) - g(X,Z)S(Y,W) \\ &\quad + g(X,W)S(Y,Z) - g(Y,W)S(X,Z); \\ \psi_2(S)(X,Y,Z,W) &= g(Y,JZ)S(X,JW) - g(X,JZ)S(Y,JW) \\ &\quad + g(X,JW)S(Y,JZ) - g(Y,JW)S(X,JZ); \\ \pi_1 = \tfrac{1}{2}\psi_1(g); \qquad \pi_2 &= \tfrac{1}{2}\psi_2(g); \qquad \pi_3 = -\psi_1(\widetilde{g}) = \psi_2(\widetilde{g}). \end{aligned} \tag{1.10}$$

It is well known that the Weyl tensor W on a $2n$-dimensional pseudo-Riemannian manifold $(n \geq 2)$ is defined as follows

$$W = R - \frac{1}{2n-2}\left\{\psi_1(\rho) - \frac{\tau}{2n-1}\pi_1\right\}. \tag{1.11}$$

Let $\alpha = \{x,y\}$ be a non-degenerate 2-plane spanned by vectors $x,y \in T_pM$, $p \in M$. The sectional curvatures of α with respect to the curvature-like tensor L are given by

$$\nu(L;p) = \frac{L(x,y,y,x)}{\pi_1(x,y,y,x)}, \qquad \nu^*(L;p) = \frac{L(x,y,y,Jx)}{\pi_1(x,y,y,x)}. \tag{1.12}$$

The square norm $\|\nabla J\|^2$ of ∇J is defined in [3] by

$$\|\nabla J\|^2 = g^{ij} g^{kl} g\left((\nabla_{e_i} J)e_k, (\nabla_{e_j} J)e_l\right). \tag{1.13}$$

Following [3], [4] we define a second square norm $\|\nabla J\|_*^2$ of ∇J with respect to the associated metric $\widetilde{g}$ by

$$\|\nabla J\|_*^2 = \widetilde{g}^{ij} \widetilde{g}^{kl} \widetilde{g}\left((\nabla_{e_i} J)e_k, (\nabla_{e_j} J)e_l\right), \tag{1.14}$$

where $\widetilde{g}^{ij} = -J_s^i g^{js}$ are the components of the inverse matrix of $\widetilde{g}$ with respect to the basis $\{e_i\}$. Then, having in mind the definition (1.2) and the properties (1.3) of the tensor F, from (1.13) and (1.14) we obtain that

$$\|\nabla J\|^2 = g^{ij} g^{kl} g^{pq} F_{ikp} F_{jlq}; \qquad \|\nabla J\|_*^2 = -\widetilde{g}^{ij} g^{kl} g^{pq} F_{ikp} F_{jlq}, \tag{1.15}$$

where $F_{ikp} = F(e_i, e_k, e_p)$.

Definition 1.1. *An almost complex manifold with Norden metric satisfying the condition* $\|\nabla J\|^2 = 0$ *is called an isotropic Kähler manifold with Norden metric.*

Definition 1.2. *An almost complex manifold with Norden metric satisfying the condition* $\|\nabla J\|^2 = \|\nabla J\|_*^2 = 0$ *is called a strong isotropic Kähler manifold with Norden metric.*

2 Complex connections and curvature tensors on conformal Kähler manifolds with Norden metric

Let (M, J, g) be a W_1-manifold with Norden metric. The Lie forms θ and $\theta^* = \theta \circ J$ are closed on M if and only if $(\nabla_X \theta) Y = (\nabla_Y \theta) X$ and $(\nabla_X \theta) JY = (\nabla_Y \theta) JX$. A W_1-manifold with closed Lie forms is called *a conformal Kähler manifold with Norden metric*. The subclass of these manifolds is denoted by W_1^0.

In [2] is introduced a cannonical linear connection (so called B-connection) D on a complex manifold with Norden metric as follows

$$D_X Y = \nabla_X Y - \frac{1}{2} J (\nabla_X J) Y. \tag{2.1}$$

It is shown that g and J are parallel with respect to the connection D. The curvature tensor K of D is proved to be Kählerian.

In [6] is studied the Yano connection ∇' given by

$$\nabla'_X Y = \nabla_X Y + \frac{1}{4} \left\{ (\nabla_X J) JY + 2 (\nabla_Y J) JX - (\nabla_{JX} J) Y \right\}.$$

It is proved that the Yano connection is torsion-free and that $\nabla' J = 0$ on a complex manifold with Norden metric. In the same paper is obtained the Kähler curvature tensor R' of type $(0,4)$ of ∇' on a W_1^0-manifold as follows

$$R' = R - \frac{1}{4n} \{\psi_1 + \psi_2\} (S) - \frac{1}{8n^2} \psi_1(M) - \frac{\theta(\Omega)}{16n^2} \{3\pi_1 + \pi_2\} + \frac{\theta(J\Omega)}{16n^2} \pi_3, \tag{2.2}$$

where

$$\begin{aligned} S(X,Y) &= (\nabla_X \theta) JY + \tfrac{1}{4n} \left[\theta(X)\theta(Y) - \theta(JX)\theta(JY)\right], \\ M(X,Y) &= \theta(X)\theta(Y) + \theta(JX)\theta(JY). \end{aligned} \tag{2.3}$$

Then, having in mind (1.7), (2.1), (2.2) and (2.3) we receive the following

Theorem 2.1. *The Kähler curvature tensors of the connections D and ∇' coincide on a conformal Kähler manifold with Norden metric, i.e. $K = R'$.*

Theorem 2.2. *Let (M, J, g) be a four-dimensional almost complex manifold with Norden metric and L be a Kähler tensor on M. Then, the tensor L has the following form*

$$L = \nu(L)\{\pi_1 - \pi_2\} + \nu^*(L)\pi_3. \tag{2.4}$$

Proof. It is known [5] that in the tangent space T_pM, $p \in M$, there exists a J-basis $\{e_1, e_2, Je_1, Je_2\}$ such that $g(e_i, e_j) = -g(Je_i, Je_j) = \delta_{ij}$, $g(e_i, Je_j) = 0$, $i, j = 1, 2$. Then, by the use of (1.9), (1.10), (1.12) and after straightforward calculations we prove the truthfulness of (2.4). □

From the last theorem and (1.8) it follows that

$$\nu(L) = \frac{\tau(L)}{8}, \qquad \nu^*(L) = \frac{\tau^*(L)}{8}. \tag{2.5}$$

Then, having in mind (2.2) and (2.3) for $n = 2$, (2.4), (2.5) and (1.8) we obtain the following

Theorem 2.3. *Let (M, J, g) be a four-dimensional W_1^0-manifold. Then, for the curvature tensor R of the Levi-Civita connection ∇ we have*

$$\begin{aligned} R = &\frac{\tau - \operatorname{div}(J\Omega)}{8}\{\pi_1 - \pi_2\} + \frac{\operatorname{tr} S^*}{16}\pi_3 - \frac{1}{8}\{\psi_1 - \psi_2\}(S) + \frac{1}{2}\left\{\psi_1(\rho) - \frac{\tau}{3}\pi_1\right\} \\ &+\frac{1}{4}\left[\frac{\operatorname{div}(J\Omega)}{2} - \frac{\tau}{3} - \frac{\theta(\Omega)}{8}\right]\pi_1, \end{aligned} \tag{2.6}$$

where $\operatorname{tr} S^* = g^{ij}S(e_i, Je_j) = -\operatorname{div}\Omega + \frac{\theta(J\Omega)}{4}$ *for* $n = 2$, $\operatorname{div}\Omega = \nabla_i\Omega^i$ *and* $\operatorname{div}(J\Omega) = \nabla_i(J_k^i\Omega^k)$.

The last theorem and (1.11) imply the following

Corollary 2.1. *Let (M, J, g) be four-dimensional W_1^0-manifold. Then, for the Weyl tensor we have*

$$W = \frac{\tau - \operatorname{div}(J\Omega)}{8}\{\pi_1 - \pi_2\} + \frac{\operatorname{tr} S^*}{16}\pi_3 - \frac{1}{8}\{\psi_1 - \psi_2\}(S) + \frac{1}{4}\left[\frac{\operatorname{div}(J\Omega)}{2} - \frac{\tau}{3} - \frac{\theta(\Omega)}{8}\right]\pi_1.$$

Next, taking into account (1.5) and (1.15) we obtain that on a W_1-manifold

$$\|\nabla J\|^2 = \frac{2}{n}\theta(\Omega), \qquad \|\nabla J\|_*^2 = -\frac{2}{n}\theta(J\Omega) \tag{2.7}$$

and from (2.7) and Theorem 2.3 we receive

Corollary 2.2. *Let (M, J, g) be a four-dimensional strong isotropic Kähler W_1^0-manifold. Then, for the curvature tensor R of ∇ we have*

$$\begin{aligned} R = &\frac{\tau - \operatorname{div}(J\Omega)}{8}\{\pi_1 - \pi_2\} - \frac{\operatorname{div}\Omega}{16}\pi_3 - \frac{1}{8}\{\psi_1 - \psi_2\}(S) + \frac{1}{2}\left\{\psi_1(\rho) - \frac{\tau}{3}\pi_1\right\} \\ &+\frac{1}{4}\left[\frac{\operatorname{div}(J\Omega)}{2} - \frac{\tau}{3}\right]\pi_1. \end{aligned}$$

3 Invariant tensors of the transformation of the Levi-Civita connections of g and $\widetilde{g}$ on a W_1-manifold

Let (M, J, g) be an almost complex manifold with Norden metric and $\widetilde{\nabla}$ be the Levi-Civita connection of the associated metric $\widetilde{g}$. In [2] is considered the tensor

$$\Phi(X, Y, Z) = g\left(\widetilde{\nabla}_X Y - \nabla_X Y, Z\right)$$

and is obtained that

$$\Phi(X, Y, Z) = \frac{1}{2}\left\{F(JZ, X, Y) - F(X, Y, JZ) - F(Y, X, JZ)\right\}. \tag{3.1}$$

By the use of (1.5) and (3.1) we receive the following

Lemma 3.1. *Let (M, J, g) be a W_1-manifold with Norden metric. Then, for the connections ∇ and $\widetilde{\nabla}$ we have*

$$\widetilde{\nabla}_X Y = \nabla_X Y + \frac{1}{2n}\left[g(X, JY)\Omega - g(X, Y)J\Omega\right]. \tag{3.2}$$

Let $\widetilde{R}$ be the curvature tensor of $\widetilde{\nabla}$. Then, having in mind (1.7) and (3.2) we obtain

Theorem 3.1. *Let (M, J, g) be a W_1-manifold with Norden metric. Then, the curvature tensors R and $\widetilde{R}$ of type $(1, 3)$ are related as follows*

$$\begin{aligned}\widetilde{R}(X, Y)Z = R(X, Y)Z &+ \tfrac{1}{2n}\left\{g(X, Z)\left[\nabla_Y J\Omega - \tfrac{1}{2n}\theta(JY)J\Omega\right]\right. \\ &-g(Y, Z)\left[\nabla_X J\Omega - \tfrac{1}{2n}\theta(JX)J\Omega\right] - g(X, JZ)\left[\nabla_Y \Omega - \tfrac{1}{2n}\theta(Y)J\Omega\right] \\ &\left.+g(Y, JZ)\left[\nabla_X \Omega - \tfrac{1}{2n}\theta(X)J\Omega\right]\right\}.\end{aligned} \tag{3.3}$$

Further, we consider the following tensors:

$$\begin{aligned}T_1(X, Y)Z &= R(X, Y)Z + \tfrac{1}{4n}\left\{g(X, Z)\left[\nabla_Y J\Omega - \tfrac{1}{2n}\theta(JY)J\Omega\right]\right. \\ &\quad -g(Y, Z)\left[\nabla_X J\Omega - \tfrac{1}{2n}\theta(JX)J\Omega\right] - g(X, JZ)\left[\nabla_Y \Omega - \tfrac{1}{2n}\theta(Y)J\Omega\right] \\ &\quad \left.+g(Y, JZ)\left[\nabla_X \Omega - \tfrac{1}{2n}\theta(X)J\Omega\right]\right\}; \\ T_2(X, Y) &= \rho(X, Y) - \tfrac{1}{2n}\left[g(X, Y)\tau - g(X, JY)\tau^*\right]; \\ T_3(X, Y) &= \left(\nabla_X \theta\right) Y + \tfrac{1}{4n}\left[g(X, Y)\theta(J\Omega) - g(X, JY)\theta(\Omega)\right].\end{aligned}$$

Then, by the use of (1.2), (1.4), (3.2), (3.3) and $\left(\nabla_X \theta\right) Y = X\theta(Y) - \theta(\nabla_X Y)$ we get the following

Theorem 3.2. *Let (M, J, g) be a W_1-manifold with Norden metric. Then, the Lie form θ and the tensors T_1, T_2, T_3 are invariant by the transformation of the connections ∇ and $\widetilde{\nabla}$.*

References

[1] G. Ganchev, A. Borisov, *Note on the almost complex manifolds with a Norden metric. Compt. Rend. Acad. Bulg. Sci., vol. 39, ℕ 5, (1986), 31-34.*

[2] G. Ganchev, K. Gribachev, V. Mihova, *B-Connections and their conformal invariants on conformally Kaehler manifolds with B-metric. Publ. Inst. Math. (Beograd) (N.S.), vol. 42 (56), (1987), 107-121.*

[3] E. Garcia-Rio, Y. Matsushita, *Isotropic Kähler structures on Engel 4-manifolds. J. Geom. Phys., vol. 33, (2000), 288-294.*

[4] K. Gribachev, M. Manev, D. Mekerov, *A Lie group as a 4-dimensional Quasi-Kähler manifold with Norden metric. JP J. Geom. Topol., (2005), (to appear).*

[5] K. Gribachev, D. Mekerov, G. Djelepov, *Generalized B-manifolds. Compt. Rend. Acad. Bulg. Sci., vol. 38,* ℕ *3, (1985), 299-302.*

[6] M. Teofilova, *Complex connections on complex manifolds with Norden metric. Contemporary Aspects of Complex Analysis, Differential Geometry and Mathematical Physics, eds. S. Dimiev and K. Sekigawa, World Sci. Publ., Singapore, (2005).*

M. Teofilova
Faculty of Mathematics and Informatics
University of Plovdiv
236 Bulgaria Blvd.
Plovdiv 4004, Bulgaria
`mar@gbg.bg`