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LIE GROUPS AS KÄHLER MANIFOLDS WITH KILLING NORDEN METRIC

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Abstract

Some properties of Lie groups and Lie algebras equipped with a bi-invariant complex structure and a pair of Killing Norden metrics are studied. An example of a Kähler manifold with Norden metric, illustrating the obtained results, is constructed on a 2-parametric family of compact Lie algebras. The curvature properties of the manifold are studied, and the form of the curvature tensor is obtained.

Key words: Kähler manifold, Norden metric, B-metric, Killing metric, bi-invariant complex structure, Lie group, Lie algebra

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1. Introduction. It is a well-known fact that on a smooth manifold equipped with an almost complex structure there exists a compatible Hermitian metric. In [1], Norden has introduced another kind of compatible metric on such manifolds, called the B-metric. The action of the almost complex structure on the tangent bundle of the manifold is an anti-isometry with respect to this metric. Norden has studied B-metric manifolds endowed with a parallel complex structure with respect to the Levi–Civita connection of the metric, i.e. Kähler manifolds with B-metric.

Almost complex manifolds with Norden metric (B-metric) are originally introduced in [2] under the term generalized B-manifolds. A classification of these

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manifolds with respect to the covariant derivative of the almost complex structure is obtained in [3], and an equivalent classification is given in [4].

Kähler manifolds with Norden metric have been recently studied by many authors, e.g. [5–8], and examples of them are given in [5, 7, 9]. In [5] these manifolds are called anti-Kählerian, and there it is shown that a compact simply connected Kähler Hermitian manifold cannot admit Kähler Norden structure.

The aim of the present paper is to study Kähler structures with Norden metric on Lie groups and their corresponding Lie algebras. In the first section we recall some necessary notions. In the second section we consider real Lie groups endowed with a left-invariant Norden metric and a special kind of left-invariant almost complex structure, called bi-invariant. We also study Lie groups equipped with a pair of ad-invariant Norden metrics. In the third section we illustrate the obtained results by an example of a 6-dimensional Kähler manifold with Norden metric. This example is constructed on a 2-parametric family of compact Lie algebras equipped with a bi-invariant complex structure and a pair of Killing Norden metrics. The Killing form of these Lie algebras is shown to be a linear combination of the Norden metrics. We study the curvature properties of the manifold, obtain the form of its curvature tensor and show that it has constant totally real sectional curvatures. We also prove that the manifold has zero Bochner tensor and obtain necessary and sufficient conditions for the manifold to be Einstein.

2. Preliminaries. Let (M, J, g) be an almost complex manifold with Norden metric, i.e. M is a $2n$-dimensional smooth manifold equipped with an almost complex structure J and a pseudo-Riemannian metric g such that

$$J^2x = -x, \qquad g(Jx, Jy) = -g(x, y) \tag{2.1}$$

for arbitrary x, y in the Lie algebra $\mathfrak{X}(M)$ of the differentiable vector fields on M.

The tensor field $\widetilde{g}$, defined by

$$\widetilde{g}(x, y) = g(x, Jy) \tag{2.2}$$

is called the associated metric of g. Because of (2.1), $\widetilde{g}$ is also a Norden metric. Both metrics are necessarily neutral, i.e. of signature (n, n).

Further, x, y, z, u will stand for arbitrary elements of $\mathfrak{X}(M)$ or vectors in the tangent space T_pM, $p \in M$.

Let ∇ be the Levi–Civita connection of g. The fundamental tensor field F of type (0,3) is defined by

$$F(x, y, z) = g((\nabla_x J)y, z) \tag{2.3}$$

and has the following properties

$$F(x, y, z) = F(x, z, y) = F(x, Jy, Jz). \tag{2.4}$$

The Nijenhuis tensor field N for J is given by

$$N(x,y) = [Jx,Jy] - [x,y] - J[Jx,y] - J[x,Jy]. \tag{2.5}$$

It is known [10] that the almost complex structure is complex if and only if it is integrable, i.e. $N = 0$.

A classification of the almost complex manifolds with Norden metric is introduced in [3]. Eight classes of these manifolds are characterized there according to the properties of F.

In the present work we focus our attention on the class of the Kähler manifolds with Norden metric which is present in all of the other classes. This class is usually denoted by $\mathcal{W}_0$, and its characteristic condition is $F = 0$ (i.e. $\nabla J = 0$).

Let R be the curvature tensor of ∇, i.e.

$$R(x,y)z = \nabla_x \nabla_y z - \nabla_y \nabla_x z - \nabla_{[x,y]} z. \tag{2.6}$$

The corresponding (0,4)-type tensor is defined by

$$R(x,y,z,u) = g\left(R(x,y)z,u\right).$$

A tensor of type (0,4) is said to be curvature-like if it has the properties of R.

Let S be a symmetric and hybrid with respect to J tensor of type (0,2), i.e. $S(x,y) = S(y,x)$ and $S(Jx,Jy) = -S(x,y)$. The essential curvature-like tensors on an almost complex manifold with Norden metric are

$$\begin{aligned}
\psi_1(S)(x,y,z,u) &= g(y,z)S(x,u) - g(x,z)S(y,u) \\
&\quad + g(x,u)S(y,z) - g(y,u)S(x,z), \\
\psi_2(S)(x,y,z,u) &= g(y,Jz)S(x,Ju) - g(x,Jz)S(y,Ju) \\
&\quad + g(x,Ju)S(y,Jz) - g(y,Ju)S(x,Jz), \\
\pi_1 = \psi_1(g), \quad \pi_2 &= \psi_2(g), \quad \pi_3 = -\psi_1(\widetilde{g}) = \psi_2(\widetilde{g}).
\end{aligned} \tag{2.7}$$

The curvature tensor R is said to be Kählerian if

$$R(x,y,Jz,Ju) = -R(x,y,z,u). \tag{2.8}$$

Because of $\nabla J = 0$ on a Kähler manifold with Norden metric the curvature tensor is Kählerian. In this case the tensor R^*, called the associated tensor of R and defined by

$$R^*(x,y,z,u) = R(x,y,z,Ju) \tag{2.9}$$

is also a curvature-like Kähler tensor.

If $\alpha = \{x, y\}$ is a non-degenerate 2-plane spanned by the vectors $x, y \in T_pM$, $p \in M$, the sectional curvatures of α are given by

$$k(\alpha; p) = \frac{R(x, y, y, x)}{\pi_1(x, y, y, x)}, \qquad k^*(\alpha; p) = \frac{R(x, y, y, Jx)}{\pi_1(x, y, y, x)}. \tag{2.10}$$

We consider the following basic sectional curvatures in T_pM with respect to the structures J and g: holomorphic sectional curvatures if $J\alpha = \alpha$, and totally real sectional curvatures if $J\alpha \perp \alpha$ with respect to g.

Let $\{e_i\}$ $(i = 1, 2, \ldots, 2n)$ be an arbitrary basis of the tangent space T_pM at a point p of M. The components of the inverse matrix of g are denoted by g^{ij} with respect to the basis $\{e_i\}$.

The Ricci tensor ρ and the scalar curvatures τ and τ^* are defined by respectively

$$\rho(y, z) = g^{ij} R(e_i, y, z, e_j), \qquad \tau = g^{ij}\rho(e_i, e_j), \qquad \tau^* = g^{ij}\rho(e_i, Je_j). \tag{2.11}$$

If R is a Kähler tensor on an almost complex manifold with Norden metric (M, J, g), $\dim M = 2n \geq 6$, the Bochner tensor $\mathcal{B}(R)$ is defined by [4]

$$\mathcal{B}(R) = R - \frac{1}{2(n-2)}\{\psi_1 - \psi_2\}(\rho) + \frac{1}{4(n-1)(n-2)}\{\tau(\pi_1 - \pi_2) + \tau^*\pi_3\}. \tag{2.12}$$

Necessary and sufficient conditions for $\mathcal{B}(R) = 0$ on a Kähler manifold with Norden metric are obtained in [3].

3. Lie groups as almost complex manifolds with Killing Norden metrics. Let G be a $2n$-dimensional real connected Lie group, and $\mathfrak{g}$ be its corresponding Lie algebra. If $\{x_1, x_2, \ldots, x_{2n}\}$ is a basis of left-invariant vector fields on G, we define a left-invariant almost complex structure J and a left-invariant Norden metric g on G as follows:

$$\begin{gathered} Jx_i = x_{i+n}, \qquad Jx_{i+n} = -x_i, \\ g(x_i, x_i) = -g(x_{i+n}, x_{i+n}) = 1, \qquad g(x_j, x_k) = 0, \quad j \neq k, \\ i = 1, 2, ..., n;\ j, k = 1, 2, ..., 2n. \end{gathered} \tag{3.1}$$

Then, (G, J, g) is an almost complex manifold with Norden metric.

Let us recall that an almost complex structure J on a Lie group G is called *bi-invariant* (*ad-invariant*) if it is invariant with respect to both, left and right translations on G. On the corresponding Lie algebra $\mathfrak{g}$ of G this condition is equivalent to $\operatorname{ad} \circ J = J \circ \operatorname{ad}$, i.e.

$$[x, Jy] = J[x, y], \qquad x, y \in \mathfrak{g}. \tag{3.2}$$

Property (3.2) implies $[Jx, Jy] = -[x, y]$. Then, obviously, if J is bi-invariant, the Nijenhuis tensor (2.5) vanishes, i.e. J is a complex structure, and (G, J, g) is a complex manifold with Norden metric.

By the help of the well-known Koszul's formula for the Levi–Civita connection of g on G, i.e. the equality

$$2g(\nabla_{x_i}x_j, x_k) = g([x_i, x_j], x_k) + g([x_k, x_i], x_j) + g([x_k, x_j], x_i), \tag{3.3}$$

we compute

$$\begin{gathered} 2g((\nabla_{x_i}J)x_j, x_k) = g([x_i, Jx_j] - J[x_i, x_j], x_k) \\ +g((J[x_k, x_i]) - [Jx_k, x_i], x_j) + g([x_k, Jx_j] - [Jx_k, x_j], x_i). \end{gathered} \tag{3.4}$$

Then, it is valid

Proposition 3.1. *Let (G, J, g) be an almost complex manifold with Norden metric defined by* (3.1)*. Then, if J is bi-invariant, the manifold (G, J, g) is Kählerian.*

Proof. The proof of the statement follows immediately by applying (3.2) in (3.4), and by taking into account the definition of F and the characteristic condition of the class of the Kähler manifolds with Norden metric. □

Further, let us recall that a left-invariant metric g on a Lie group G is called *ad-invariant* if $g(\mathrm{ad}x(y), z) = -g(\mathrm{ad}x(z), y)$, i.e.

$$g([x, y], z) = -g([x, z], y) \tag{3.5}$$

for arbitrary $x, y, z \in \mathfrak{g}$. A metric satisfying (3.5) is also called Killing.

Proposition 3.2. *Let G be a real connected Lie group equipped with a left-invariant almost complex structure J and a pair of left-invariant Norden metrics g and $\widetilde{g}$, defined by* (2.1) *and* (2.2)*. Then, any two of the following conditions imply the third one*:

(i) *J is bi-invariant;*

(ii) *g is Killing;*

(iii) *$\widetilde{g}$ is Killing.*

Proof. Let us prove that (i) and (ii) imply (iii). Let J be a bi-invariant complex structure, and g be a Killing Norden metric on G. Then, by combining (3.2) and (3.5) we get

$$g(J[x, y], z) = -g(J[x, z], y), \tag{3.6}$$

which is equivalent to $\widetilde{g}([x, y], z) = -\widetilde{g}([x, z], y)$. The last equality is a necessary and sufficient condition for $\widetilde{g}$ defined by (2.2) to be Killing. The truthfulness of the other two statements is proved analogously. □

The Killing form B on a Lie algebra $\mathfrak{g}$ is defined by

$$B(x, y) = \mathrm{tr}(\mathrm{ad}x \circ \mathrm{ad}y). \tag{3.7}$$

According to Cartan's criterion B is non-degenerate iff $\mathfrak{g}$ is semi-simple.

If J is a bi-invariant complex structure on $\mathfrak{g}$, then by (3.7) and (3.2) it is easy to see that $B(Jx, Jy) = -B(x, y)$. Hence, we have

Proposition 3.3. *Let $\mathfrak{g}$ be a semi-simple Lie algebra equipped with a bi-invariant complex structure J. Then, the Killing form B of $\mathfrak{g}$ is a Killing Norden metric.*

4. A 6-dimensional example. In this section we construct and study an example of a 6-dimensional Kähler manifold with Norden metric on a 2-parametric family of compact Lie algebras admitting a bi-invariant complex structure and a pair of Killing Norden metrics.

Let $\mathfrak{g}$ be a 6-dimensional Lie algebra corresponding to a real connected Lie group G. If $\{x_1, x_2, ..., x_6\}$ is a basis of $\mathfrak{g}$ and $[x_i, x_j] = C_{ij}^k x_k$, the structural constants C_{ij}^k satisfy the anti-commutativity condition $C_{ij}^k = -C_{ji}^k$ and the Jacobi identity

$$C_{ij}^k C_{ks}^l + C_{js}^k C_{ki}^l + C_{si}^k C_{kj}^l = 0. \tag{4.1}$$

We define a left-invariant almost complex structure J and a left-invariant Norden metric on G analogously to (3.1), i.e.

$$\begin{aligned} &Jx_1 = x_4, \qquad &Jx_2 = x_5, \qquad &Jx_3 = x_6, \\ &Jx_4 = -x_1, \qquad &Jx_2 = -x_5, \qquad &Jx_6 = -x_3 \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} &g(x_1, x_1) = g(x_2, x_2) = g(x_3, x_3) \\ &= -g(x_4, x_4) = -g(x_5, x_5) = -g(x_6, x_6) = 1, \\ &g(x_i, x_j) = 0, \quad i \neq j, \quad i, j = 1, 2, ..., 6. \end{aligned} \tag{4.3}$$

Then the associated metric $\widetilde{g}$ defined by (2.2) has the non-zero components

$$\widetilde{g}(x_1, x_4) = \widetilde{g}(x_2, x_5) = \widetilde{g}(x_3, x_6) = -1. \tag{4.4}$$

The condition (3.2) for J to be bi-invariant implies

$$\begin{gathered} [x_i, x_j] = -[x_{i+3}, x_{j+3}], \qquad [x_i, x_{j+3}] = -[x_j, x_{i+3}] = J[x_i, x_j], \\ [x_i, x_{i+n}] = 0 \end{gathered} \tag{4.5}$$

for $i < j$; $i, j = 1, 2, 3$. Then (4.5) and the Jacobi identity yield the following

Proposition 4.1. *Let (G, J, g) be a 6-dimensional almost complex manifold with Norden metric defined by* (4.2) *and* (4.3)*. Then (G, J, g) is a Kähler manifold* *admitting a bi-invariant complex structure iff the Lie algebra $\mathfrak{g}$ of G belongs to the 12-parametric family of Lie algebras with non-zero commutators*

$$
\begin{aligned}
&[x_1,x_2]=-[x_4,x_5]=\lambda_1x_1+\lambda_2x_2+\lambda_3x_3+\lambda_4x_4+\lambda_5x_5+\lambda_6x_6,\\
&[x_1,x_3]=-[x_4,x_6]=\lambda_7x_1+\lambda_8x_2-\lambda_2x_3+\lambda_9x_4+\lambda_{10}x_5-\lambda_5x_6,\\
&[x_1,x_5]=-[x_2,x_4]=-\lambda_4x_1-\lambda_5x_2-\lambda_6x_3+\lambda_1x_4+\lambda_2x_5+\lambda_3x_6,\\
&[x_1,x_6]=-[x_3,x_4]=-\lambda_9x_1-\lambda_{10}x_2+\lambda_5x_3+\lambda_7x_4+\lambda_8x_5-\lambda_2x_6,\\
&[x_2,x_3]=-[x_5,x_6]=\lambda_{11}x_1-\lambda_7x_2+\lambda_1x_3+\lambda_{12}x_4-\lambda_9x_5+\lambda_4x_6,\\
&[x_2,x_6]=-[x_3,x_5]=-\lambda_{12}x_1+\lambda_9x_2-\lambda_4x_3+\lambda_{11}x_4-\lambda_7x_5+\lambda_1x_6,
\end{aligned}
\tag{4.6}
$$

where $\lambda_i\in\mathbb{R}$, $i=1,2,\dots,12$.

Next, we consider the Kähler manifold (G,J,g) with corresponding Lie algebra defined by (4.6) and ask for the metric g to be Killing. Conditions (3.5), (4.3) and (4.6) imply $\lambda_{11}=-\lambda_8=\lambda_3$, $\lambda_{12}=-\lambda_{10}=\lambda_6$, and the rest $\lambda_i=0$.

We put $\lambda=\lambda_3$ and $\mu=\lambda_6$. Then, it is valid

Proposition 4.2. *Let (G,J,g) be a 6-dimensional almost complex manifold with Norden metric defined by (4.2) and (4.3). Then (G,J,g) is a Kähler manifold equipped with a bi-invariant complex structure and a Killing Norden metric iff the Lie algebra $\mathfrak{g}$ of G belongs to the 2-parametric family of simple Lie algebras with non-zero commutators*

$$
\begin{aligned}
&[x_1,x_2]=-[x_4,x_5]=\lambda x_3+\mu x_6,\\
&[x_1,x_3]=-[x_4,x_6]=-\lambda x_2-\mu x_5,\\
&[x_1,x_5]=-[x_2,x_4]=-\mu x_3+\lambda x_6,\\
&[x_1,x_6]=-[x_3,x_4]=\mu x_2-\lambda x_5,\\
&[x_2,x_3]=-[x_5,x_6]=\lambda x_1+\mu x_4,\\
&[x_2,x_6]=-[x_3,x_5]=-\mu x_1+\lambda x_4.
\end{aligned}
\tag{4.7}
$$

Let us remark that according to Proposition 3.2 the associated Norden metric $\widetilde{g}$ of g is also Killing.

The Killing form B of the Lie algebras (4.7) is determined by

$$
B=4\begin{pmatrix}
\mu^2-\lambda^2 & 0 & 0 & 2\lambda\mu & 0 & 0\\
0 & \mu^2-\lambda^2 & 0 & 0 & 2\lambda\mu & 0\\
0 & 0 & \mu^2-\lambda^2 & 0 & 0 & 2\lambda\mu\\
2\lambda\mu & 0 & 0 & \lambda^2-\mu^2 & 0 & 0\\
0 & 2\lambda\mu & 0 & 0 & \lambda^2-\mu^2 & 0\\
0 & 0 & 2\lambda\mu & 0 & 0 & \lambda^2-\mu^2
\end{pmatrix}.
\tag{4.8}
$$

Then, $\det B=-4096(\lambda^2+\mu^2)^6$, i.e. the Killing form is negative for all $\lambda,\mu\in\mathbb{R}$, $(\lambda,\mu)\neq(0,0)$. Hence the Lie algebras defined by (4.7) are compact.

Furthermore, from (4.3), (4.4) and (4.8) it follows that the Killing form B can be represented as a linear combination of the Killing Norden metrics g and $\widetilde{g}$ as follows:

$$B = -4\left\{(\lambda^2 - \mu^2)g + 2\lambda\mu\widetilde{g}\right\}. \tag{4.9}$$

Next, we study the curvature properties of the Kähler manifold (G, J, g), where the Lie algebra of G is defined by (4.7).

Let us recall that on a Lie algebra equipped with a Killing metric g the following identities are valid for the components of the Levi–Civita connection ∇, the curvature tensor R and the Ricci tensor ρ of g:

$$\nabla_x y = \frac{1}{2}[x, y], \quad R(x, y, z, u) = -\frac{1}{4}g([x, y], [z, u]), \quad \rho(x, y) = -\frac{1}{4}B(x, y). \tag{4.10}$$

Then by the help of (4.10) we compute the essential non-zero components $R_{ijkl} = R(x_i, x_j, x_k, x_l)$ of the curvature tensor R

$$\begin{aligned}
&R_{1221} = R_{1331} = -R_{1551} = -R_{1661} = R_{2332}\\
&= -R_{2442} = -R_{2662} = -R_{3443} = -R_{3553} = R_{4554}\\
&= R_{4664} = R_{5665} = R_{1245} = R_{1346} = R_{2356}\\
&= -R_{1524} = -R_{1634} = -R_{2635} = \tfrac{1}{4}(\lambda^2 - \mu^2),\\
&R_{1251} = R_{1361} = R_{2142} = R_{2362} = R_{3143} = R_{3253} = -R_{4254}\\
&= -R_{4364} = -R_{5145} = -R_{5365} = -R_{6146} = -R_{6256} = -\tfrac{1}{2}\lambda\mu.
\end{aligned} \tag{4.11}$$

Also, by (4.10) and (4.8) we get the following essential non-zero components $\rho_{ij} = \rho(x_i, x_j)$ of the Ricci tensor:

$$\begin{aligned}
&\rho_{11} = \rho_{22} = \rho_{33} = -\rho_{44} = -\rho_{55} = -\rho_{66} = \lambda^2 - \mu^2,\\
&\rho_{14} = \rho_{25} = \rho_{36} = -2\lambda\mu.
\end{aligned} \tag{4.12}$$

According to (2.11) and (4.12) we compute the scalar curvatures of the manifold

$$\tau = 6(\lambda^2 - \mu^2), \qquad \tau^* = -12\lambda\mu. \tag{4.13}$$

Let us consider the characteristic 2-planes α_{ij} spanned by the basic vectors $\{x_i, x_j\}$: totally real 2-planes – $\alpha_{12}, \alpha_{13}, \alpha_{15}, \alpha_{16}, \alpha_{23}, \alpha_{24}, \alpha_{26}, \alpha_{34}, \alpha_{35}, \alpha_{45}, \alpha_{46}, \alpha_{56}$

and holomorphic 2-planes $\alpha_{14}, \alpha_{25}, \alpha_{36}$. By (2.10) and (4.11) we obtain

$$
\begin{aligned}
&k(\alpha_{12}) = k(\alpha_{13}) = k(\alpha_{15}) = k(\alpha_{16}) = k(\alpha_{23}) \\
&= k(\alpha_{24}) = k(\alpha_{26}) = k(\alpha_{34}) = k(\alpha_{35}) = k(\alpha_{45}) \\
&= k(\alpha_{46}) = k(\alpha_{56}) = \tfrac{1}{4}(\lambda^2 - \mu^2), \\
&k^*(\alpha_{12}) = k^*(\alpha_{13}) = k^*(\alpha_{15}) = k^*(\alpha_{16}) = k^*(\alpha_{23}) \\
&= k^*(\alpha_{24}) = k^*(\alpha_{26}) = k^*(\alpha_{34}) = k^*(\alpha_{35}) = k^*(\alpha_{45}) \\
&= k^*(\alpha_{46}) = k^*(\alpha_{56}) = -\tfrac{1}{2}\lambda\mu, \\
&k(\alpha_{14}) = k(\alpha_{25}) = k(\alpha_{36}) = k^*(\alpha_{14}) = k^*(\alpha_{25}) = k^*(\alpha_{36}) = 0.
\end{aligned} \tag{4.14}
$$

Hence, the manifold is of constant totally real sectional curvatures k and k^*.

It has been proved [6] that a Kähler manifold with Norden metric (M, J, g) $(\dim M \geq 4)$ is of pointwise constant totally real sectional curvatures ν and ν^*, i.e. $k(\alpha; p) = \nu(p)$, $k^*(\alpha; p) = \nu^*(p)$ for any non-degenerate 2-plane α in T_pM, $p \in M$, iff

$$R = \nu\{\pi_1 - \pi_2\} + \nu^*\pi_3. \tag{4.15}$$

Both functions ν and ν^* are absolute constants (i.e. they do not depend on the point p) if the manifold is connected and $\dim M \geq 6$.

Then, taking into account (4.14), we have

Proposition 4.3. *The curvature tensor of the* 6*-dimensional Kähler manifold* (G, J, g)*, defined by* (4.2), (4.3) *and* (4.7)*, has the form (4.15).*

It is easy to prove that if R satisfies (4.15), the manifold is Bochner-flat, i.e. $\mathcal{B}(R) = 0$.

It is known that if $\rho = ag + b\widetilde{g}$, $a, b = \text{const.}$, on an almost complex manifold with Norden metric, the manifold is called almost Einstein. If $\rho = ag$ the manifold is Einstein. Thus, (4.10) and (4.9) yield

Proposition 4.4. *The* 6*-dimensional Kähler manifold* (G, J, g)*, defined by* (4.2), (4.3) *and* (4.7)*, is almost Einstein.*

Corollary 4.1. *The manifold* (G, J, g) *is Einstein iff* $\lambda = 0, \mu \neq 0$ *or* $\mu = 0, \lambda \neq 0$.

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