On a Class Almost Contact Manifolds with Norden Metric

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Abstract

Certain curvature properties and scalar invariants of the manifolds belonging to one of the main classes almost contact manifolds with Norden metric are considered. An example illustrating the obtained results is given and studied.
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Introduction

The geometry of the almost contact manifolds with Norden metric (B-metric) is a natural extension of the geometry of the almost complex manifolds with Norden metric (B-metric) in the odd dimensional case.

Almost contact manifolds with Norden metric are introduced in [1]. Eleven basic classes of these manifolds are characterized there according to the properties of the covariant derivatives of the almost contact structure.

In this work we focus our attention on one of the basic classes almost contact manifolds with Norden metric, namely the class $\mathcal{F}_{11}$. We study some curvature properties and relations between certain scalar invariants of the manifolds belonging to this class. In the last section we illustrate the obtained results by constructing and studying an example of an $\mathcal{F}_{11}$-manifold on a Lie group.

1 Preliminaries

Let M be a $(2n+1)$-dimensional smooth manifold, and let (φ, ξ, η) be an almost contact structure on M, i.e. φ is an endomorphism of the tangent

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bundle of M, ξ is a vector field, and η is its dual 1-form such that

$$\varphi^2 = -\mathrm{Id} + \eta \otimes \xi, \qquad \eta(\xi) = 1. \tag{1.1}$$

Then, (M, φ, ξ, η) is called an *almost contact manifold*.

We equip (M, φ, ξ, η) with a compatible pseudo-Riemannian metric g satisfying

$$g(\varphi x, \varphi y) = -g(x, y) + \eta(x)\eta(y) \tag{1.2}$$

for arbitrary x, y in the Lie algebra $\mathfrak{X}(M)$ of the smooth vector fields on M. Then, g is called a *Norden metric* (*B-metric*), and $(M, \varphi, \xi, \eta, g)$ is called an *almost contact manifold with Norden metric*.

From (1.1), (1.2) it follows $\varphi\xi = 0$, $\eta \circ \varphi = 0$, $\eta(x) = g(x, \xi)$, $g(\varphi x, y) = g(x, \varphi y)$.

The associated metric $\tilde{g}$ of g is defined by $\tilde{g}(x, y) = g(x, \varphi y) + \eta(x)\eta(y)$ and is a Norden metric, too. Both metrics are necessarily of signature $(n + 1, n)$.

Further, x, y, z, u will stand for arbitrary vector fields in $\mathfrak{X}(M)$.

Let ∇ be the Levi-Civita connection of g. The fundamental tensor F of type (0,3) is defined by

$$F(x, y, z) = g\left((\nabla_x \varphi) y, z\right) \tag{1.3}$$

and has the properties

$$\begin{aligned} &F(x, y, z) = F(x, z, y), \\ &F(x, \varphi y, \varphi z) = F(x, y, z) - F(x, \xi, z)\eta(y) - F(x, y, \xi)\eta(z). \end{aligned} \tag{1.4}$$

From the last equation and $\varphi\xi = 0$ it follows $F(x, \xi, \xi) = 0$.

Let $\{e_i, \xi\}$ $(i = 1, 2, ..., 2n)$ be a basis of the tangent space T_pM at an arbitrary point p of M, and g^{ij} be the components of the inverse matrix of (g_{ij}) with respect to $\{e_i, \xi\}$. The following 1-forms are associated with F:

$$\begin{aligned} &\theta(x) = g^{ij} F(e_i, e_j, x), && \theta^*(x) = g^{ij} F(e_i, \varphi e_j, x), \\ &\omega(x) = F(\xi, \xi, x), && \omega^* = \omega \circ \varphi. \end{aligned} \tag{1.5}$$

We denote by Ω the vector field corresponding to ω, i.e. $\omega(x) = g(x, \Omega)$.

The Nijenhuis tensor N of the almost contact structure (φ, ξ, η) is defined by [6] $N(x, y) = [\varphi, \varphi](x, y) + d\eta(x, y)\xi$, i.e.

$$N(x, y) = \varphi^2[x, y] + [\varphi x, \varphi y] - \varphi[\varphi x, y] - \varphi[x, \varphi y] + (\nabla_x \eta)y.\xi - (\nabla_y \eta)x.\xi$$

In terms of the covariant derivatives of φ and η the tensor N is expressed as follows

$$\begin{aligned} N(x, y) = &(\nabla_{\varphi x} \varphi) y - (\nabla_{\varphi y} \varphi) x - \varphi(\nabla_x \varphi) y + \varphi(\nabla_y \varphi) x \\ &+ (\nabla_x \eta) y.\xi - (\nabla_y \eta) x.\xi, \end{aligned} \tag{1.6}$$

where $(\nabla_x \eta)y = F(x, \varphi y, \xi)$.

The almost contact structure in said to be integrable if $N = 0$. In this case the almost contact manifold is called *normal* [6].

A classification of the almost contact manifolds with Norden metric is introduced in [1]. This classification consists of eleven basic classes $\mathcal{F}_i$ $(i = 1, 2, ..., 11)$ characterized according to the properties of F. The special class $\mathcal{F}_0$ of the φ-Kähler-type almost contact manifolds with Norden metric is given by the condition $F = 0$ $(\nabla\varphi = \nabla\xi = \nabla\eta = 0)$. The classes for which F is expressed explicitly by the other structural tensors are called *main classes*.

In the present work we focus our attention on one of the main classes of these manifolds, namely the class $\mathcal{F}_{11}$, which is defined by the characteristic condition [1]

$$F(x, y, z) = \eta(x)\{\eta(y)\omega(z) + \eta(z)\omega(y)\}. \tag{1.7}$$

By (1.5) and (1.7) we get that on a $\mathcal{F}_{11}$-manifold $\theta = \omega$, $\theta^* = 0$. We also have

$$(\nabla_x \omega^*)y = (\nabla_x \omega)\varphi y + \eta(x)\eta(y)\omega(\Omega). \tag{1.8}$$

The 1-forms ω and ω^* are said to be closed if $\mathrm{d}\omega = \mathrm{d}\omega^* = 0$. Since ∇ is symmetric, necessary and sufficient conditions for ω and ω^* to be closed are

$$(\nabla_x \omega)y = (\nabla_y \omega)x, \qquad (\nabla_x \omega)\varphi y = (\nabla_y \omega)\varphi x. \tag{1.9}$$

The curvature tensor R of ∇ is defined as usually by

$$R(x, y)z = \nabla_x \nabla_y z - \nabla_y \nabla_x z - \nabla_{[x,y]} z, \tag{1.10}$$

and its corresponding tensor of type (0,4) is given by $R(x, y, z, u) = g(R(x, y)z, u)$. The Ricci tensor ρ and the scalar curvatures τ and τ^* are defined by, respectively

$$\rho(y, z) = g^{ij} R(e_i, y, z, e_j), \qquad \tau = g^{ij}\rho(e_i, e_j), \qquad \tau^* = g^{ij}\rho(e_i, \varphi e_j). \tag{1.11}$$

The tensor R is said to be of φ-Kähler-type if

$$R(x, y, \varphi z, \varphi u) = -R(x, y, z, u). \tag{1.12}$$

Let $\alpha = \{x, y\}$ be a non-degenerate 2-section spanned by the vectors $x, y \in T_pM$, $p \in M$. The sectional curvature of α is defined by

$$k(\alpha; p) = \frac{R(x, y, y, x)}{\pi_1(x, y, y, x)}, \tag{1.13}$$

where $\pi_1(x, y, z, u) = g(y, z)g(x, u) - g(x, z)g(y, u)$.

In [5] there are introduced the following special sections in T_pM: a ξ-section if $\alpha = \{x, \xi\}$, a φ-holomorphic section if $\varphi\alpha = \alpha$ and a totally real section if $\varphi\alpha \perp \alpha$ with respect to g.

The square norms of $\nabla\varphi$, $\nabla\eta$ and $\nabla\xi$ are defined by, respectively [3]:

$$\begin{aligned} &||\nabla\varphi||^2 = g^{ij}g^{ks}g\left((\nabla_{e_i}\varphi)e_k, (\nabla_{e_j}\varphi)e_s\right), \\ &||\nabla\eta||^2 = ||\nabla\xi||^2 = g^{ij}g^{ks}(\nabla_{e_i}\eta)e_k(\nabla_{e_j}\eta)e_s. \end{aligned} \tag{1.14}$$

We introduce the notion of an isotropic Kähler-type almost contact manifold with Norden metric analogously to [2].

Definition 1.1. An almost contact manifold with Norden metric is called *isotropic Kählerian* if $||\nabla\varphi||^2 = ||\nabla\eta||^2 = 0$ ($||\nabla\varphi||^2 = ||\nabla\xi||^2 = 0$).

2 Curvature properties of $\mathcal{F}_{11}$-manifolds

In this section we obtain relations between certain scalar invariants on $\mathcal{F}_{11}$-manifolds with Norden metric and give necessary and sufficient conditions for such manifolds to be isotropic Käherian.

First, by help of (1.3), (1.4), (1.6), (1.7), (1.14) and direct computation we obtain

Proposition 2.1. *On a $\mathcal{F}_{11}$-manifold it is valid*

$$||\nabla\varphi||^2 = -||N||^2 = -2||\nabla\eta||^2 = 2\omega(\Omega). \tag{2.1}$$

Then, (2.1) and Definition 1.1 yield

Corollary 2.1. *On a $\mathcal{F}_{11}$-manifold the following conditions are equivalent*:

(i) *the manifold is isotropic Kählerian*;

(ii) *the vector Ω is isotopic, i.e. $\omega(\Omega) = 0$*;

(iii) *the Nijenhuis tensor N is isotropic.*

It is known that the almost contact structure satisfies the Ricci identity, i.e.

$$\begin{aligned} &(\nabla_x\nabla_y\varphi)z - (\nabla_y\nabla_x\varphi)z = R(x,y)\varphi z - \varphi R(x,y)z, \\ &(\nabla_x\nabla_y\eta)z - (\nabla_y\nabla_x\eta)z = -\eta(R(x,y)z). \end{aligned} \tag{2.2}$$

Then, taking into account the definitions of φ, F, and $\nabla g = 0$, the equalities (2.2) imply

$$\begin{aligned} &(\nabla_xF)(y,z,\varphi u) - (\nabla_yF)(x,z,\varphi u) = R(x,y,z,u) \\ &\qquad +R(x,y,\varphi z,\varphi u) - R(x,y,z,\xi)\eta(u), \end{aligned} \tag{2.3}$$

$$(2.4)\qquad (\nabla_x F)(y,\varphi z,\xi)-(\nabla_y F)(x,\varphi z,\xi)=-R(x,y,z,\xi).$$

By (1.3), (2.3) and (2.4) we get

$$(2.5)\qquad R(x,y,\varphi z,\varphi u)=-R(x,y,z,u)+\psi_4(S)(x,y,z,u),$$

where the tensor $\psi_4(S)$ is defined by [4]

$$(2.6)\qquad \begin{aligned}\psi_4(S)(x,y,z,u)&=\eta(y)\eta(z)S(x,u)-\eta(x)\eta(z)S(y,u)\\&+\eta(x)\eta(u)S(y,z)-\eta(y)\eta(u)S(x,z).\end{aligned}$$

and

$$(2.7)\qquad S(x,y)=(\nabla_x\omega)\varphi y-\omega(\varphi x)\omega(\varphi y).$$

Then, the following holds

Proposition 2.2. *On a $\mathcal{F}_{11}$-manifold we have*

$$\tau+\tau^{**}=2\mathrm{div}(\varphi\Omega)=2\rho(\xi,\xi),$$

where $\tau^{**}=g^{is}g^{jk}R(e_i,e_j,\varphi e_k,\varphi e_s)$.

Proof. The truthfulness of the statement follows from (1.11) and (2.5) by straightforward computation. □

Having in mind (1.12) and (2.5), we conclude that the curvature tensor on a $\mathcal{F}_{11}$-manifold is of φ-Käher-type if and only if $\psi_4(S)=0$. Because of (2.6) the last condition holds true iff $S=0$. Then, taking into account (1.8) and (2.7) we prove

Proposition 2.3. *The curvature tensor of a $\mathcal{F}_{11}$-manifold with Norden metric is φ-Käherian iff*

$$(2.8)\qquad (\nabla_x\omega^*)y=\eta(x)\eta(y)\omega(\Omega)+\omega^*(x)\omega^*(y).$$

The condition (2.8) implies $\mathrm{d}\omega^*=0$, i.e. ω^* is closed.

3 An example

In this section we present and study a $(2n+1)$-dimensional example of a $\mathcal{F}_{11}$-manifold constructed on a Lie group.

Let G be a $(2n+1)$-dimensional real connected Lie group, and $\mathfrak{g}$ be its corresponding Lie algebra. If $\{x_0,x_1,...,x_{2n}\}$ is a basis of left-invariant vector fields on G, we define a left-invariant almost contact structure (φ,ξ,η) by

$$(3.1)\qquad \begin{array}{llll}\varphi x_i=x_{i+n}, & \varphi x_{i+n}=-x_i, & \varphi x_0=0, & i=1,2,...,n,\\ \xi=x_0, & \eta(x_0)=1, & \eta(x_j)=0, & j=1,2,...,2n.\end{array}$$

We also define a left-invariant pseudo-Riemannian metric g on G by

$$
\begin{aligned}
&g(x_0,x_0)=g(x_i,x_i)=-g(x_{i+n},x_{i+n})=1, \quad i=1,2,...,n,\\
&g(x_j,x_k)=0, \quad j\neq k, \quad j,k=0,1,,...,2n.
\end{aligned} \tag{3.2}
$$

Then, according to (1.1) and (1.2), (G,φ,ξ,η,g) is an almost contact manifold with Norden metric.

Let the Lie algebra $\mathfrak{g}$ of G be given by the following non-zero commutators

$$
[x_i,x_0]=\lambda_i x_0, \qquad i=1,2,...,2n, \tag{3.3}
$$

where $\lambda_i\in\mathbb{R}$. Equalities (3.3) determine a $2n$-parametric family of solvable Lie algebras.

Further, we study the manifold (G,φ,ξ,η,g) with Lie algebra $\mathfrak{g}$ defined by (3.3). The well-known Koszul's formula for the Levi-Civita connection of g on G, i.e. the equality

$$
2g(\nabla_{x_i}x_j,x_k)=g([x_i,x_j],x_k)+g([x_k,x_i],x_j)+g([x_k,x_j],x_i),
$$

implies the following components of the Levi-Civita connection:

$$
\begin{aligned}
&\nabla_{x_i}x_j=\nabla_{x_i}\xi=0, \qquad \nabla_\xi x_i=-\lambda_i\xi, \quad i,j=1,2,...,2n,\\
&\nabla_\xi\xi=\textstyle\sum_{k=1}^{n}(\lambda_k x_k-\lambda_{k+n}x_{k+n}).
\end{aligned} \tag{3.4}
$$

Then, by (1.3) and (3.4) we obtain the essential non-zero components of F:

$$
F(\xi,\xi,x_i)=\omega(x_i)=-\lambda_{i+n}, \qquad F(\xi,\xi,x_{i+n})=\omega(x_{i+n})=\lambda_i, \tag{3.5}
$$

for $i=1,2,...,n$. Hence, by (1.7) and (3.5) we have

Proposition 3.1. *The almost contact manifold with Norden metric (G,φ,ξ,η,g) defined by (3.1), (3.2) and (3.3) belongs to the class $\mathcal{F}_{11}$.*

Moreover, by (1.9), (3.4) and (3.5) we establish that the considered manifold has closed 1-forms ω and ω^*.

Taking into account (3.4) and (1.10) we obtain the essential non-zero components of the curvature tensor as follows

$$
R(x_i,\xi,\xi,x_j)=-\lambda_i\lambda_j, \quad i,j=1,2,...,2n. \tag{3.6}
$$

By (3.6) it follows that $R(x_i,x_j,\varphi x_k,\varphi x_s)=0$ for all $i,j,k,s=0,1,...,2n$. Then, according to (2.5) and (2.7) we get

Proposition 3.2. *The curvature tensor and the Ricci tensor of the $\mathcal{F}_{11}$-manifold (G,φ,ξ,η,g) defined by (3.1), (3.2) and (3.3) have the form, respectively*

$$
R=\psi_4(S), \qquad \rho(x,y)=\eta(x)\eta(y)\mathrm{tr}S+S(x,y),
$$

where S is defined by (2.7) and $\mathrm{tr}S=\mathrm{div}(\varphi\Omega)$.

We compute the essential non-zero components of the Ricci tensor as follows

$$
\begin{aligned}
&\rho(x_i, x_j) = -\lambda_i \lambda_j, \quad i = 1, 2, ..., 2n, \\
&\rho(\xi, \xi) = -\textstyle\sum_{k=1}^{n} \left(\lambda_k^2 - \lambda_{k+n}^2\right).
\end{aligned} \tag{3.7}
$$

By (1.11) and (3.7) we obtain the curvatures of the considered manifold

$$
\tau = -2 \sum_{k=1}^{n} \left(\lambda_k^2 - \lambda_{k+n}^2\right), \qquad \tau^* = -2 \sum_{k=1}^{n} \lambda_k \lambda_{k+n}. \tag{3.8}
$$

Let us consider the characteristic 2-sections α_{ij} spanned by the vectors $\{x_i, x_j\}$: ξ-sections $\alpha_{0,i}$ $(i = 1, 2, ..., 2n)$, φ-holomorphic sections $\alpha_{i,i+n}$ $(i = 1, 2, ..., n)$, and the rest are totally real sections. Then, by (1.13), (3.1) and (3.6) it follows

Proposition 3.3. *The $\mathcal{F}_{11}$-manifold with Norden metric $(G, \varphi, \xi, \eta, g)$ defined by (3.1), (3.2) and (3.3) has zero totally real and φ-holomorphic sectional curvatures, and its ξ-sectional curvatures are given by*

$$
k(\alpha_{0,i}) = -\frac{\lambda_i^2}{g(x_i, x_i)}, \qquad i = 1, 2, ..., 2n.
$$

By (3.2) and (3.5) we obtain the corresponding vector Ω to ω and its square norm

$$
\Omega = -\sum_{k=1}^{n} \left(\lambda_{k+n} x_k + \lambda_k x_{k+n}\right), \qquad \omega(\Omega) = -\sum_{k=1}^{n} \left(\lambda_k^2 - \lambda_{k+n}^2\right). \tag{3.9}
$$

Then, by (3.9) and Corollary 2.1 we prove

Proposition 3.4. *The $\mathcal{F}_{11}$-manifold with Norden metric $(G, \varphi, \xi, \eta, g)$ defined by (3.1), (3.2) and (3.3) is isotropic Kählerian iff the condition $\sum_{k=1}^{n} \left(\lambda_k^2 - \lambda_{k+n}^2\right) = 0$ holds.*

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