COMPLEX CONNECTIONS ON COMPLEX MANIFOLDS WITH NORDEN METRIC

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The class of the complex manifolds with Norden metric is considered. The Yano connection is introduced. The properties of the curvature tensor and the Bochner tensor of the Yano connection are studied.

1 Introduction

Let (M, J) be an almost complex manifold with an almost complex structure J. It is well known that on such manifolds there exists Hermitian metric g with the property $g(JX, JY) = g(X, Y)$ for any $X, Y \in \mathfrak{X}(M)$. In this case the manifold (M, J, g) is called an almost Hermitian manifold. On the other hand, according to A. P. Norden [11], on an almost complex manifold there exists also an indefinite metric g such that $g(JX, JY) = -g(X, Y)$, $X, Y \in \mathfrak{X}(M)$. Almost complex manifolds with such a metric are originally introduced in [9] under the name generalized B-manifolds and the metric g is called B-metric. Later, a classification of almost complex manifolds with B-metric is given in [5] and equivalent characteristic conditions for each of these classes are obtained in [6]. From another point of view, these manifolds are studied in [8] where they are called almost complex Riemmanian manifolds. Further, in this paper such manifolds are called almost complex manifolds with Norden metric.

Furthermore, these manifolds are considered by many authors, for example [1], [2], [3], [4]. Let us note that examples of three of the main classes of almost complex manifolds with Norden metric are given in [3].

An important problem in the geometry of almost complex manifolds with Norden metric is the existing of linear connections with respect to which the almost complex structure is parallel. In [6] there is introduced a B-connection with non-zero torsion tensor field on a complex manifold with Norden metric (M, J, g) with respect to which g and J are parallel. In the same paper there is proved that the Bochner curvature tensor of the B-connection is an invariant with respect to special conformal transformations of the metric g.

In this paper we introduce and study the Yano connection on an almost complex manifold with Norden metric.

2 Preliminaries

Let (M, J) be a $2n$-dimensional almost complex manifold, $J^2 = -id$. A metric g on M is called *Norden* if the almost complex structure J is an antiisometry of the tangent space at any point of M, i.e.

$$g(JX, JY) = -g(X, Y), \quad X, Y \in \mathfrak{X}(M).$$

Then, the manifold (M, J, g) is called *an almost complex manifold with Norden metric*.

The associated metric $\widetilde{g}$ of the manifold is defined by

$$\widetilde{g}(X, Y) = g(JX, Y) = g(X, JY).$$

Obviously, the metric $\widetilde{g}$ is also a Norden metric. Both metrics are necessarily of signature (n, n).

Let ∇ be the Levi-Civita connection of the metric g. The tensor field F of type $(0, 3)$ on the manifold is defined by

$$F(X, Y, Z) = g\left((\nabla_X J)Y, Z\right). \tag{1}$$

This tensor has the following symmetries:

$$F(X, Y, Z) = F(X, Z, Y) = F(X, JY, JZ). \tag{2}$$

The associated Lee 1-forms θ and $\widetilde{\theta}$ on M are given by $\theta(x) = g^{ij}F(e_i, e_j, x)$, $\widetilde{\theta} = \theta \circ J$, where x is a tangent vector at an arbitrary point $p \in M$, $\{e_i\}_{i=1,\dots,2n}$ is a basis of the tangent space T_pM and $\left(g^{ij}\right)$ is the inverse of the matrix associated to g.

The Nijenhuis tensor field N of the manifold is defined as follows

$$N(X, Y) = (\nabla_X J)\, JY - (\nabla_Y J)\, JX + (\nabla_{JX} J)\, Y - (\nabla_{JY} J)\, X. \tag{3}$$

In [5] the eight classes of almost complex manifolds with Norden metric are characterized by conditions for F as follows

1. The class W_0 of the Kähler manifolds with Norden metric:

$$F = 0 \iff \nabla J = 0$$

2. The class W_1:

$$\begin{aligned} F(X, Y, Z) = \tfrac{1}{2n} & \left[g(X, Y)\theta(Z) + g(X, Z)\theta(Y)\right. \\ & \left. + g(X, JY)\theta(JZ) + g(X, JZ)\theta(JY)\right] \end{aligned} \tag{4}$$

3. The class W_2 of the special complex manifolds with Norden metric:

$$F(X,Y,JZ)+F(Y,Z,JX)+F(Z,X,JY)=0, \quad \theta=0 \iff N=0, \quad \theta=0 \tag{5}$$

4. The class W_3 of quasi-Kähler manifolds with Norden metric:

$$F(X,Y,Z)+F(Y,Z,X)+F(Z,X,Y)=0$$

5. The class $W_1 \oplus W_2$ of the complex manifolds with Norden metric:

$$F(X,Y,JZ)+F(Y,Z,JX)+F(Z,X,JY)=0 \iff N=0 \tag{6}$$

6. The class $W_2 \oplus W_3$ of semi-Kähler manifolds with Norden metric:

$$\theta=0$$

7. The class $W_1 \oplus W_3$:

$$\begin{aligned}&F(X,Y,Z)+F(Y,Z,X)+F(Z,X,Y)=\\&\tfrac{1}{n}\left[g(X,Y)\theta(Z)+g(X,Z)\theta(Y)+g(Y,Z)\theta(X)\right.\\&\left.+g(X,JY)\theta(JZ)+g(X,JZ)\theta(JY)+g(Y,JZ)\theta(JX)\right]\end{aligned}$$

8. The class of almost complex manifolds with Norden metric: no conditions.

In the paper [6] there are considered two types of conformal transformations of the Norden metric g on an almost complex manifold (M,J,g):

1. Conformal transformations of type I

$$\overline{g}=e^{2u}g,$$

where u is a pluriharmonic function on M.

2. Conformal transformations of type II

$$\overline{g}=e^{2u}\left(\cos 2v g+\sin 2v\widetilde{g}\right),$$

where $u+iv$ is a holomorphic function on M, i.e. $dv=-du\circ J$.

It is proved that the subclass W_1^0 of W_1 with closed Lee 1-forms θ and $\widetilde{\theta}$ is conformally equivalent to a Kähler manifold with Norden metric by a transformation of type I. The manifolds belonging to W_1^0 are called *conformal Kähler manifolds with Norden metric*. The class W_1^0 is closed with respect to the conformal transformations of type I and type II.

Let us note that examples of some of the classes are given as follows: in [1] for W_2 and $W_2 \oplus W_3$; in [3] for W_0, W_1 and W_2; in [4] for W_0 and W_1^0; in [6] for W_1^0; in [7] for W_0.

A tensor L of type $(0,4)$ is called *a curvature-like tensor* if it satisfies the following conditions for any $X, Y, Z, W \in \mathfrak{X}(M)$:

$$\begin{aligned}
&L(X,Y,Z,W) = -L(Y,X,Z,W);\\
&L(X,Y,Z,W) + L(Y,Z,X,W) + L(Z,X,Y,W) = 0;\\
&L(X,Y,Z,W) = -L(X,Y,W,Z).
\end{aligned}$$

A curvature-like tensor L is called *a Kähler tensor* if it satisfies the condition

$$L(X,Y,JZ,JW) = -L(X,Y,Z,W), \qquad X,Y,Z,W \in \mathfrak{X}(M).$$

Then, the associated tensor $\widetilde{L}$ defined by $\widetilde{L}(X,Y,Z,W) = L(X,Y,Z,JW)$ is also a Kähler tensor.

Let us consider the following tensors of type $(0,4)$, where S is a tensor of type $(0,2)$:

$$\begin{aligned}
\psi_1(S)(X,Y,Z,W) &= g(Y,Z)S(X,W) - g(X,Z)S(Y,W)\\
&\quad + g(X,W)S(Y,Z) - g(Y,W)S(X,Z);\\
\psi_2(S)(X,Y,Z,W) &= g(Y,JZ)S(X,JW) - g(X,JZ)S(Y,JW)\\
&\quad + g(X,JW)S(Y,JZ) - g(Y,JW)S(X,JZ);\\
\pi_1(X,Y,Z,W) &= \tfrac{1}{2}\psi_1(g)(X,Y,Z,W)\\
&= g(Y,Z)g(X,W) - g(X,Z)g(Y,W); \qquad (7)\\
\pi_2(X,Y,Z,W) &= \tfrac{1}{2}\psi_2(g)(X,Y,Z,W)\\
&= g(Y,JZ)g(X,JW) - g(X,JZ)g(Y,JW);\\
\pi_3(X,Y,Z,W) &= -\psi_1(\widetilde{g})(X,Y,Z,W) = \psi_2(\widetilde{g})(X,Y,Z,W)\\
&= -g(Y,Z)g(X,JW) + g(X,Z)g(Y,JW)\\
&\quad - g(X,W)g(Y,JZ) + g(Y,W)g(X,JZ).
\end{aligned}$$

It is known [6] that the tensor $\psi_1(S)$ is a curvature-like tensor iff S is symmetric and the tensor $\psi_2(S)$ is a curvature-like tensor iff S is symmetric and hybrid with respect to J, i.e. $S(JX,Y) = S(JY,X)$. In this case the tensors $\pi_1 - \pi_2$, π_3 and $\psi_1(S) - \psi_2(S)$ are Kähler tensors.

Let L be a Kähler tensor over T_pM, $p \in M$ and $\{e_i\}_{i=1,\dots,2n}$ be a basis of T_pM. Then the Ricci tensor $\rho(L)$ and the scalar curvatures $\tau(L)$ and $\widetilde{\tau}(L)$

are given by

$$\begin{aligned} &\rho\,(L)\,(Y,Z) = g^{is} L(e_i, Y, Z, e_s); \\ &\tau(L) = g^{jk} \rho\,(L)\,(e_j, e_k); \\ &\widetilde{\tau}(L) = \tau(\widetilde{L}) = g^{jk} \rho\,(L)\,(e_j, Je_k). \end{aligned} \tag{8}$$

The associated Bochner curvature tensor $B(L)$ is defined by

$$\begin{aligned} B(L) = L &- \tfrac{1}{2(n-2)} \left\{ \psi_1(\rho) - \psi_2(\rho) \right\} \\ &+ \tfrac{1}{4(n-1)(n-2)} \left\{ \tau(\pi_1 - \pi_2) + \widetilde{\tau}\pi_3 \right\}, \quad n \geq 3. \end{aligned} \tag{9}$$

In [6] there is introduced the B-connection D on $(M, J, g) \in W_1$. It is proved that if K is the Kähler curvature tensor for D then the Bochner tensor $B(K)$ is a conformal invariant of type I and type II.

3 Curvature properties of W_1-manifolds

Let (M, J, g) be a W_1-manifold. Then, having in mind (6), the Nijenhuis tensor vanishes on M. The Lee 1-forms θ and $\widetilde{\theta}$ are said to be closed iff $d\theta = d\widetilde{\theta} = 0$ or the following equivalent conditions hold:

$$(\nabla_X \theta)\, Y = (\nabla_Y \theta)\, X, \quad (\nabla_X \widetilde{\theta}) Y = (\nabla_Y \widetilde{\theta}) X. \tag{10}$$

Taking into account (1), (4) and (10) we obtain the following

Lemma 1 *If $(M, J, g) \in W_1^0$ then the following conditions are valid:*

$$(\nabla_X \theta)\, Y = (\nabla_Y \theta)\, X, \quad (\nabla_X \theta)\, JY = (\nabla_Y \theta)\, JX,$$

$$(\nabla_X J)\, Y = \frac{1}{2n} \left[g(X, Y)\Omega + g(X, JY)J\Omega + \theta(Y)X + \theta(JY)JX \right], \tag{11}$$

where Ω is the Lee vector corresponding to θ, i.e. $g(X, \Omega) = \theta(X)$.

Let R be the curvature tensor of ∇, i.e. $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$. The corresponding tensor of type $(0, 4)$ is denoted by the same letter and is given by $R(X, Y, Z, W) = g\,(R(X, Y)Z, W)$. Then Lemma 1, (1) and (4) imply the following conditions

$$(\nabla_X F)\, (Y, Z, W) = g\, ((\nabla_X K)\, (Y, Z), W)\, , \tag{12}$$

$$R(X, Y)JZ = JR(X, Y)Z + (\nabla_X K)\, (Y, Z) - (\nabla_Y K)\, (X, Z), \tag{13}$$

where $K(X, Y) = \frac{1}{2n} \left[g(X, Y)\Omega + g(X, JY)J\Omega + \theta(Y)X + \theta(JY)JX \right]$.

Now let us consider the following tensors of type $(0,2)$:

$$\begin{array}{l} S(X,Y) = (\nabla_X\theta)\,JY + \frac{1}{4n}\left[\theta(X)\theta(Y) - \theta(JX)\theta(JY)\right], \\ M(X,Y) = \theta(X)\theta(Y) + \theta(JX)\theta(JY). \end{array} \tag{14}$$

They have the following symmetries

$$S(JX,JY) = -S(X,Y), \quad M(JX,JY) = M(X,Y).$$

Theorem 2 *Let (M,J,g) be a conformal Kähler manifold with Norden metric. Then the curvature tensor R of ∇ has the following property*

$$\begin{array}{l} R(X,Y,JZ,JW) = -R(X,Y,Z,W) \\ +\frac{1}{2n}\left\{\left[\psi_1+\psi_2\right](S) + \frac{1}{4n}\left[\psi_1+\psi_2\right](M) + \frac{1}{2n}\theta(\Omega)\left[\pi_1+\pi_2\right]\right\}(X,Y,Z,W). \end{array} \tag{15}$$

Proof. Having in mind (12), the condition (13) implies

$$R(X,Y,JZ,W) = R(X,Y,Z,JW) + (\nabla_X F)(Y,Z,W) - (\nabla_Y F)(X,Z,W). \tag{16}$$

Then, taking into account Lemma 1, (4), (7), (14) from (16) we receive (15). ■

Next, we define the tensor field R^* of type $(0,4)$ by

$$R^* = R - \frac{1}{2n}\psi_1(L), \tag{17}$$

where

$$L = S + \frac{1}{4n}M + \frac{\theta(\Omega)}{4n}g. \tag{18}$$

Since the tensor L is symmetric then R^* is a curvature-like tensor on any W_1^0-manifold. Moreover, taking into account Theorem 2, (7), (17) and (18) we obtain $R^*(X,Y,JZ,JW) = -R^*(X,Y,Z,W)$, i.e. R^* is a Kähler tensor.

Then, according to (7), (8) and (17) we get the following interconnections between the corresponding Ricci tensors and the scalar curvatures of R and R^*:

$$\begin{array}{l} \rho^* = \rho - \frac{1}{2n}\left[g\mathrm{tr}L + 2(n-1)L\right]; \\ \tau^* = \tau - \frac{2n-1}{n}\mathrm{tr}L, \quad \mathrm{tr}L = \frac{n}{2n-1}(\tau - \tau^*). \end{array}$$

Hence we obtain

$$L(Y,Z) = \frac{n}{n-1}\left\{\rho(Y,Z) - \rho^*(Y,Z) - \frac{\tau - \tau^*}{2(2n-1)}g(Y,Z)\right\}.$$

The last equality and (17) imply

$$R^* - \frac{1}{2(n-1)}\left\{\psi_1(\rho^*) - \frac{\tau^*}{2n-1}\pi_1\right\} = R - \frac{1}{2(n-1)}\left\{\psi_1(\rho) - \frac{\tau}{2n-1}\pi_1\right\}. \tag{19}$$

The Weyl tensor $W(R)$ of R is defined as follows

$$W(R) = R - \frac{1}{n-2}\left\{\psi_1(\rho) - \frac{\tau}{n-1}\pi_1\right\}. \tag{20}$$

It is well known that the Weyl tensor of type $(0,4)$ of R is an invariant of the conformal transformation of type I, i.e. $W(\overline{R}) = e^{2u}W(R)$. Then, using (19) and (20) we obtain the following

Theorem 3 *Let (M,J,g) be a conformal Kähler manifold with Norden metric. Then the Weyl tensors of R and R^* coincide, i.e. $W(R) = W(R^*)$.*

4 The Yano connection on almost complex manifolds with Norden metric

Let (M,J,g) be an almost complex manifold with Norden metric. Following [12] and [13] we consider *the Yano connection* defined by

$$\nabla'_X Y = \nabla_X Y + T(X,Y), \tag{21}$$

where

$$T(X,Y) = \frac{1}{4}\left[(\nabla_X J)\, JY + 2\,(\nabla_Y J)\, JX - (\nabla_{JX} J)\, Y\right]. \tag{22}$$

The torsion tensor field Q of ∇' is given by

$$Q(X,Y) = \nabla'_X Y - \nabla'_Y X - [X,Y] = T(X,Y) - T(Y,X). \tag{23}$$

Taking into account (3), (22) and (23) we receive the following

Lemma 4 *Let (M,J,g) be an almost complex manifold with Norden metric. Then the Yano connection is symmetric iff the Nijenhuis tensor field vanishes on M.*

Let us note that the Yano connection is symmetric on the classes W_1, W_2 and $W_1 \oplus W_2$ according to Lemma 4 and the conditions (4), (5) and (6).

Theorem 5 *Let (M,J,g) be an almost complex manifold with Norden metric and ∇' be the Yano connection on M. Then $\nabla' J = 0$ iff $N = 0$.*

Proof. The well known equality $(\nabla'_X J)\,Y = \nabla'_X JY - J\nabla'_X Y$ and (3), (21), (22) imply

$$(\nabla'_X J)\,Y = -\frac{1}{2}N(X, JY).$$

Thus, the vanishing of $\nabla' J$ is equivalent to the vanishing of N. ■

Next, we consider the Yano connection on W_1-manifolds. The conditions (11) and (22) imply

$$\begin{aligned} T(X,Y) = \tfrac{1}{4n}\,[&g(X,JY)\Omega - g(X,Y)J\Omega + \theta(JX)Y - \theta(X)JY \\ &+\theta(JY)X - \theta(Y)JX]\,. \end{aligned} \tag{24}$$

Theorem 6 *Let (M,J,g) be a W_1-manifold. Then the covariant derivatives of g and $\widetilde{g}$ with respect to the Yano connection satisfy the following conditions:*

$$(\nabla'_X g)\,(Y,Z) = \frac{1}{2n}\,[\theta(X)g(Y,JZ) - \theta(JX)g(Y,Z)]\,; \tag{25}$$

$$(\nabla'_X \widetilde{g})\,(Y,Z) = -\frac{1}{2n}\,[\theta(X)g(Y,Z) + \theta(JX)g(Y,JZ)]\,. \tag{26}$$

Proof. From (1), (2), (6), (21) and (22) we obtain:

$$(\nabla'_X g)\,(Y,Z) = \frac{1}{2}\,[2F(Y,X,JZ) + F(JX,Y,Z) - F(X,Y,JZ)]\,; \tag{27}$$

$$(\nabla'_X \widetilde{g})\,(Y,Z) = \frac{1}{2}\,[F(JZ,X,JY) - F(Z,X,Y)]\,. \tag{28}$$

Then, taking into account (4), the equalities (27) and (28) imply (25) and (26), respectively. ■

Let R' be the curvature tensor of ∇' of type $(1,3)$. Then, according to (21) we have:

$$\begin{aligned} R'(X,Y)Z = R(X,Y)Z &+ (\nabla_X T)\,(Y,Z) - (\nabla_Y T)\,(X,Z) \\ &+T\,(X,T(Y,Z)) - T\,(Y,T(X,Z))\,; \end{aligned}$$

$$\begin{aligned} R'(X,Y,Z,W) = R(X,Y,Z,W) &+ (\nabla_X T)\,(Y,Z,W) - (\nabla_Y T)\,(X,Z,W) \\ &+T\,(X,T(Y,Z),W) - T\,(Y,T(X,Z),W)\,, \end{aligned} \tag{29}$$

where $R'(X,Y,Z,W) = g\,(R'(X,Y)Z,W)$ and

$$\begin{aligned} T(X,Y,Z) = \tfrac{1}{4n}\,[&g(X,JY)\theta(Z) - g(X,Y)\theta(JZ) + g(X,Z)\theta(JY) \\ &-g(X,JZ)\theta(Y) + g(Y,Z)\theta(JX) - g(Y,JZ)\theta(X)]\,. \end{aligned} \tag{30}$$

Theorem 7 *Let (M,J,g) be a conformal Kähler manifold with Norden metric. Then the curvature tensors R and R' are related as follows*

$$R' = R - \tfrac{1}{4n}\left\{[\psi_1+\psi_2](S) + \tfrac{1}{2n}\psi_1(M) + \tfrac{1}{4n}\theta(\Omega)[3\pi_1+\pi_2] - \tfrac{1}{4n}\theta(J\Omega)\pi_3\right\}. \tag{31}$$

Proof. By the use of Lemma 1, (4), (7), (14), (30) and after straightforward calculations in the right side of (29) we receive (31). ∎

The last theorem, (7) and (15) imply

Corollary 8 *Let (M,J,g) be a conformal Kähler manifold with Norden metric. Then the curvature tensor of the Yano connection is Kählerian.*

Theorem 9 *Let (M,J,g) be a conformal Kähler manifold with Norden metric. Then the Bochner curvature tensors of the Kähler tensors R' and R^* coincide.*

Proof. From (17), (18) and (31) we get

$$R' = R^* + \frac{1}{4n}[\psi_1-\psi_2](A), \tag{32}$$

where

$$A = S + \frac{\theta(\Omega)}{8n}g - \frac{\theta(J\Omega)}{8n}\widetilde{g}.$$

Then, for the corresponding Ricci tensors $\rho' = \rho(R')$, $\rho^* = \rho(R^*)$ and the scalar curvatures $\tau' = \tau(R')$, $\tau^* = \tau(R^*)$, $\widetilde{\tau}' = \widetilde{\tau}(R')$, $\widetilde{\tau}^* = \widetilde{\tau}(R^*)$ we obtain

$$\rho' = \rho^* + \frac{\tau'-\tau^*}{4(n-1)}g - \frac{\widetilde{\tau}'-\widetilde{\tau}^*}{4(n-1)}\widetilde{g} + \frac{n-2}{2n}A.$$

From the last equality, (9) and (32) it follows $B(R') = B(R^*)$. ∎

Lemma 10 *Let (M,J,g) be a conformal Kähler manifold with Norden metric and let $(M,J,\overline{g})$ be its conformally equivalent complex manifold with Norden metric by a transformation of type I. Then the corresponding curvature tensors R and $\overline{R}$ are related as follows*

$$\overline{R} = e^{2u}\left\{R - \psi_1(G) - \pi_1\sigma(U)\right\},$$

where $G(X,Y) = (\nabla_X\sigma)Y - \sigma(X)\sigma(Y)$, $\sigma(X) = Xu = du(X)$, $U = grad\ u$.

Taking into account the last lemma and the definition of the tensor R^*, we obtain the following interconnection of $(1,3)$-tensors

$$\overline{R} = R^* + \frac{\theta(\Omega)}{4n^2}\pi_3. \tag{33}$$

Having in mind (9), from (33) we receive the following

Corollary 11 *The Bochner tensors of $\overline{R}$ and R^* are coincident on a conformal Kähler manifold with Norden metric.*

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