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On a Class Almost Contact Manifolds with Norden Metric

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Let $(M, \varphi, \xi, \eta, g)$ be a $(2n+1)$-dimensional *almost contact manifold with Norden metric* (*B-metric*), i.e. (φ, ξ, η) is an *almost contact structure*, and g is a pseudo-Riemannian metric, called a *Norden metric* (*B-metric*) such that [1]

$$\varphi^2 x = -x + \eta(x)\xi, \qquad \eta(\xi) = 1,$$
$$g(\varphi x, \varphi y) = -g(x, y) + \eta(x)\eta(y).$$

The associated metric $\tilde{g}$ of g is defined by

$$\tilde{g}(x, y) = g(x, \varphi y) + \eta(x)\eta(y)$$

and is a Norden metric, too. Both metrics are necessarily of signature $(n+1, n)$.

Let ∇ be the Levi-Civita connection of g. The fundamental tensor F is defined by

$$F(x,y,z) = g\left((\nabla_x\varphi)y, z\right)$$

and has the following properties

$$F(x,y,z) = F(x,z,y),$$
$$F(x,\varphi y,\varphi z) = F(x,y,z) - F(x,\xi,z)\eta(y) - F(x,y,\xi)\eta(z).$$

The following 1-forms are associated with F:

$$\theta(x) = g^{ij}F(e_i,e_j,x), \qquad \theta^*(x) = g^{ij}F(e_i,\varphi e_j,x),$$
$$\omega(x) = F(\xi,\xi,x), \qquad \omega^* = \omega\circ\varphi.$$

The corresponding vector field to ω is denoted by Ω, i.e. $\omega(x) = g(x,\Omega)$.

The Nijenhuis tensor N of the almost contact structure (φ, ξ, η) is defined by [6]

$$N(x,y) = \varphi^2[x,y] + [\varphi x, \varphi y] - \varphi[\varphi x, y] - \varphi[x, \varphi y]$$

$$+ (\nabla_x \eta)y.\xi - (\nabla_y \eta)x.\xi$$

The almost contact structure in said to be integrable if $N = 0$. In this case the almost contact manifold is called *normal* [6].

A classification of the almost contact manifolds with Norden metric is introduced in [1]. Eleven basic classes $\mathcal{F}_i$ $(i = 1, 2, ..., 11)$ are characterized there according to the properties of F.

The classes for which F is expressed explicitly by the other structural tensors are called *main classes.*

In the present work we focus our attention on the class $\mathcal{F}_{11}$ given by

$$\mathcal{F}_{11}: \quad F(x, y, z) = \eta(x)\{\eta(y)\omega(z) + \eta(z)\omega(y)\}.$$

The curvature tensor R of ∇ is defined as usually by

$$R(x,y)z = \nabla_x \nabla_y z - \nabla_y \nabla_x z - \nabla_{[x,y]} z,$$

and its corresponding tensor of type (0,4) is given by

$$R(x,y,z,u) = g(R(x,y)z,u).$$

The Ricci tensor ρ and the scalar curvatures τ and τ^* are defined by, respectively

$$\rho(y,z) = g^{ij} R(e_i, y, z, e_j), \qquad \tau = g^{ij} \rho(e_i, e_j), \qquad \tau^* = g^{ij} \rho(e_i, \varphi e_j).$$

R is called a *φ-Kähler-type tensor* if

$$R(x,y,\varphi z, \varphi u) = -R(x,y,z,u).$$

Let $\alpha = \{x, y\}$ be a non-degenerate 2-section spanned by the vectors $x, y \in T_pM$, $p \in M$. The sectional curvature of α is defined by

$$k(\alpha; p) = \frac{R(x, y, y, x)}{\pi_1(x, y, y, x)},$$

where $\pi_1(x, y, z, u) = g(y, z)g(x, u) - g(x, z)g(y, u)$.

In [5] there are introduced the following special sections in T_pM:

- a ξ-section if $\alpha = \{x, \xi\}$;
- a φ-holomorphic section if $\varphi\alpha = \alpha$;
- a totally real section if $\varphi\alpha \perp \alpha$ with respect to g.

The square norms of $\nabla\varphi$, $\nabla\eta$ and $\nabla\xi$ are defined by, respectively [3]:

$$||\nabla\varphi||^2 = g^{ij}g^{ks}g\left((\nabla_{e_i}\varphi)e_k, (\nabla_{e_j}\varphi)e_s\right),$$

$$||\nabla\eta||^2 = ||\nabla\xi||^2 = g^{ij}g^{ks}(\nabla_{e_i}\eta)e_k(\nabla_{e_j}\eta)e_s.$$

Definition 1.1. An almost contact manifold with Norden metric is called *isotropic Kählerian* if

$$||\nabla\varphi||^2 = ||\nabla\eta||^2 = 0 \qquad (||\nabla\varphi||^2 = ||\nabla\xi||^2 = 0).$$

Proposition 2.1. On a $\mathcal{F}_{11}$-manifold it is valid

$$||\nabla\varphi||^2 = -||N||^2 = -2||\nabla\eta||^2 = 2\omega(\Omega).$$

Corollary 2.1. On a $\mathcal{F}_{11}$-manifold the following conditions are equivalent:

(i) the manifold is isotropic Kählerian;

(ii) the vector Ω is isotopic, i.e. $\omega(\Omega) = 0$;

(iii) the Nijenhuis tensor N is isotropic.

Proposition 2.2. On a $\mathcal{F}_{11}$-manifold we have

$$\tau + \tau^{**} = 2\mathrm{div}(\varphi\Omega) = 2\rho(\xi, \xi),$$

where $\tau^{**} = g^{is}g^{jk}R(e_i, e_j, \varphi e_k, \varphi e_s)$.

Proposition 2.3 The curvature tensor of a $\mathcal{F}_{11}$-manifold with Norden metric satisfies

$$R(x, y, \varphi z, \varphi u) = -R(x, y, z, u) + \psi_4(S)(x, y, z, u),$$

where the tensor $\psi_4(S)$ is defined by [4]

$$\begin{aligned}\psi_4(S)(x, y, z, u) &= \eta(y)\eta(z)S(x, u) - \eta(x)\eta(z)S(y, u)\\ &+ \eta(x)\eta(u)S(y, z) - \eta(y)\eta(u)S(x, z).\end{aligned}$$

$$S(x, y) = (\nabla_x\omega)\varphi y - \omega(\varphi x)\omega(\varphi y).$$

Proposition 2.4. The curvature tensor of a $\mathfrak{F}_{11}$-manifold with Norden metric is φ-Käherian iff

$$(\nabla_x \omega^*)y = \eta(x)\eta(y)\omega(\Omega) + \omega^*(x)\omega^*(y),$$

where $\omega^* = \omega \circ \varphi$.

Let G be a $(2n+1)$-dimensional real connected Lie group, and $\mathfrak{g}$ be its corresponding Lie algebra. If $\{x_0, x_1, ..., x_{2n}\}$ is a basis of left-invariant vector fields on G, we define a left-invariant almost contact structure (φ, ξ, η) by

$$\varphi x_i = x_{i+n}, \quad \varphi x_{i+n} = -x_i, \quad \varphi x_0 = 0, \quad i = 1, 2, ..., n,$$

$$\xi = x_0, \quad \eta(x_0) = 1, \quad \eta(x_j) = 0, \quad j = 1, 2, ..., 2n.$$

We also define a left-invariant pseudo-Riemannian metric g on G by

$$g(x_0, x_0) = g(x_i, x_i) = -g(x_{i+n}, x_{i+n}) = 1, \quad i = 1, 2, ..., n,$$

$$g(x_j, x_k) = 0, \quad j \neq k, \quad j, k = 0, 1, , ..., 2n.$$

Then, $(G, \varphi, \xi, \eta, g)$ is an almost contact manifold with Norden metric.

Let the Lie algebra $\mathfrak{g}$ of G be given by the following non-zero commutators

$$[x_i, x_0] = \lambda_i x_0, \qquad i = 1, 2, ..., 2n, \tag{3.1}$$

where $\lambda_i \in \mathbb{R}$.

Equalities (3.1) determine a $2n$-parametric family of solvable Lie algebras.

Further, we study the manifold $(G, \varphi, \xi, \eta, g)$ with Lie algebra $\mathfrak{g}$ defined by (3.1).

The components of the Levi-Civita connection:

$$\nabla_{x_i} x_j = \nabla_{x_i} \xi = 0, \qquad \nabla_\xi x_i = -\lambda_i \xi, \quad i, j = 1, 2, ..., 2n,$$

$$\nabla_\xi \xi = \textstyle\sum_{k=1}^{n} (\lambda_k x_k - \lambda_{k+n} x_{k+n}).$$

The essential non-zero components of F:

$$F(\xi, \xi, x_i) = \omega(x_i) = -\lambda_{i+n}, \qquad F(\xi, \xi, x_{i+n}) = \omega(x_{i+n}) = \lambda_i,$$
$$i = 1, 2, ..., n.$$

Proposition 3.1. The almost contact manifold with Norden metric $(G, \varphi, \xi, \eta, g)$ defined by (3.1) belongs to the class $\mathcal{F}_{11}$.

Remark 3.1. The considered manifold has closed 1-forms ω and ω^*.

Curvature properties

- The components of the curvature tensor R:

$$R(x_i, \xi, \xi, x_j) = -\lambda_i \lambda_j, \qquad i, j = 1, 2, ..., 2n.$$

Because of $R(x_i, x_j, \varphi x_k, \varphi x_s) = 0$ for all $i, j, k, s = 0, 1, ..., 2n$ and Proposition 2.3 we obtain

Proposition 3.2. The curvature tensor and the Ricci tensor of the $\mathcal{F}_{11}$-manifold $(G, \varphi, \xi, \eta, g)$ defined by (3.1) have the form, respectively

$$R = \psi_4(S), \qquad \rho(x, y) = \eta(x)\eta(y)\mathrm{tr}S + S(x, y),$$

where $S(x, y) = (\nabla_x \omega)\varphi y - \omega(\varphi x)\omega(\varphi y)$ and $\mathrm{tr}S = \mathrm{div}(\varphi\Omega)$.

- The essential components of the Ricci tensor ρ:

$$\rho(x_i, x_j) = -\lambda_i\lambda_j, \quad i = 1, 2, ..., 2n,$$

$$\rho(\xi, \xi) = -\textstyle\sum_{k=1}^{n}\left(\lambda_k^2 - \lambda_{k+n}^2\right).$$

- The scalar curvatures τ and τ^*:

$$\tau = -2\sum_{k=1}^{n}\left(\lambda_k^2 - \lambda_{k+n}^2\right), \qquad \tau^* = -2\sum_{k=1}^{n}\lambda_k\lambda_{k+n}.$$

• Sectional curvatures:

The characteristic 2-sections α_{ij} spanned by the vectors $\{x_i, x_j\}$ are the following:

ξ-sections $\alpha_{0,i}$ $(i = 1, 2, ..., 2n)$

φ-holomorphic sections $\alpha_{i,i+n}$ $(i = 1, 2, ..., n)$

the rest α_{ij} are totally real sections.

Proposition 3.3. The $\mathcal{F}_{11}$-manifold with Norden metric $(G, \varphi, \xi, \eta, g)$ defined by (3.1) has zero totally real and φ-holomorphic sectional curvatures, and its ξ-sectional curvatures are given by

$$k(\alpha_{0,i}) = -\frac{\lambda_i^2}{g(x_i, x_i)}, \qquad i = 1, 2, ..., 2n.$$

Isotropic Käher properties

- The vector field Ω corresponding to the 1-form ω:

$$\Omega = -\sum_{k=1}^{n} (\lambda_{k+n} x_k + \lambda_k x_{k+n}), \qquad \omega(\Omega) = -\sum_{k=1}^{n} \left(\lambda_k^2 - \lambda_{k+n}^2\right).$$

Proposition 3.4. The $\mathcal{F}_{11}$-manifold with Norden metric $(G, \varphi, \xi, \eta, g)$ defined by (3.1) is isotropic Kählerian iff the condition

$$\sum_{k=1}^{n} \left(\lambda_k^2 - \lambda_{k+n}^2\right) = 0$$

holds.

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