10th International Workshop on Complex Structures, Integrability and Vector Fields

September 13-17, 2010, Sofia

Complex Connections on Conformal Kähler Manifolds with Norden Metric

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* This work is supported by Project MU-5 of The Fund for Scientific Research of The University of Plovdiv

1. Preliminaries

Let (M, J, g) be a $2n$-dimensional *almost complex manifold with Norden metric*, i.e. J is an *almost complex structure* and g is a *pseudo-Riemannian metric* on M such that

$$J^2x = -x, \qquad g(Jx, Jy) = -g(x, y), \qquad x, y \in \mathfrak{X}(M).$$

The *associated metric* $\widetilde{g}$ of g is given by

$$\widetilde{g}(x, y) = g(x, Jy)$$

and is a Norden metric, too. Both metrics are necessarily of signature (n, n).

Let ∇ be the Levi-Civita connection of the metric g. The fundamental tensor field F of type $(0,3)$ on M is defined by

$$F(x,y,z) = g\left((\nabla_x J)y, z\right)$$

and has the following symmetries

$$F(x,y,z) = F(x,z,y) = F(x,Jy,Jz).$$

The Lie 1-forms θ and θ^* associated with F, and the Lie vector Ω, corresponding to θ, are defined by, respectively

$$\theta(x) = g^{ij}F(e_i,e_j,x), \qquad \theta^* = \theta \circ J, \qquad \theta(x) = g(x,\Omega),$$

where $\{e_i\}$ $(i = 1, 2, \ldots, 2n)$ is an arbitrary basis of T_pM at a point p of M, and g^{ij} are the components of the inverse matrix of g with respect to the basis $\{e_i\}$.

A classification of the almost complex manifolds with Norden metric is introduced by G. Ganchev and A. Borisov in [1]*, where eight classes of these manifolds are characterized according to the properties of F. The three basic classes $\mathcal{W}_i$ $(i = 1, 2, 3)$ are given by, respectively

- the class $\mathcal{W}_{\mathbf{1}}$:

$$F(x,y,z) = \frac{1}{2n}\left[g(x,y)\theta(z) + g(x,Jy)\theta(Jz) + g(x,z)\theta(y) + g(x,Jz)\theta(Jy)\right];$$

- the class $\mathcal{W}_{\mathbf{2}}$ of *the special complex manifolds with Norden metric*:

$$F(x,y,Jz) + F(y,z,Jx) + F(z,x,Jy) = 0, \quad \theta = 0;$$

*[1] G. Ganchev, A. Borisov, *Note on the almost complex manifolds with a Norden metric*, Compt. Rend. Acad. Bulg. Sci. 39(5) (1986), 31–34.

• the class $\mathcal{W}_3$ of *the quasi-Kähler manifolds with Norden metric*:

$$F(x,y,z)+F(y,z,x)+F(z,x,y)=0.$$

The special class $\mathcal{W}_0$ of *the Kähler manifolds with Norden metric* is characterized by the condition $F=0$ and is contained in each of the other classes.

Let R be the curvature tensor of ∇, i.e.

$$R(x,y)z=\nabla_x\nabla_y z-\nabla_y\nabla_x z-\nabla_{[x,y]}z.$$

The corresponding (0,4)-type tensor is defined by

$$R(x,y,z,u)=g\left(R(x,y)z,u\right).$$

A tensor L of type (0,4) is said to be *curvature-like* if it has the properties of R, i.e.

$$L(x,y,z,u) = -L(y,x,z,u) = -L(x,y,u,z),$$
$$L(x,y,z,u) + L(y,z,x,u) + L(z,x,y,u) = 0.$$

The Ricci tensor $\rho(L)$ and the scalar curvatures $\tau(L)$ and $\tau^*(L)$ of L are defined by:

$$\rho(L)(y,z) = g^{ij} L(e_i, y, z, e_j),$$

$$\tau(L) = g^{ij}\rho(L)(e_i, e_j), \qquad \tau^*(L) = g^{ij}\rho(L)(e_i, Je_j).$$

A curvature-like tensor L is called *a Kähler tensor* if

$$L(x,y,Jz,Ju) = -L(x,y,z,u).$$

Let S be a tensor of type (0,2). We consider the following tensors [3]*:

$$\begin{aligned}\psi_1(S)(x,y,z,u) &= g(y,z)S(x,u) - g(x,z)S(y,u)\\ &\quad + g(x,u)S(y,z) - g(y,u)S(x,z),\end{aligned}$$

$$\begin{aligned}\psi_2(S)(x,y,z,u) &= g(y,Jz)S(x,Ju) - g(x,Jz)S(y,Ju)\\ &\quad + g(x,Ju)S(y,Jz) - g(y,Ju)S(x,Jz),\end{aligned}$$

$$\pi_1 = \tfrac{1}{2}\psi_1(g), \qquad \pi_2 = \tfrac{1}{2}\psi_2(g), \qquad \pi_3 = -\psi_1(\widetilde{g}) = \psi_2(\widetilde{g}).$$

$\psi_1(S)$ is curvature-like if S is symmetric;
$\psi_2(S)$ is curvature-like is S is symmetric and hybrid with respect to J, i.e. $S(x,Jy) = S(y,Jx)$;
In the last case the tensor $\{\psi_1 - \psi_2\}(S)$ is Kählerian.

*[3] G. Ganchev, K. Gribachev, V. Mihova, *B-connections and their conformal invariants on conformally Kähler manifolds with B-metric*, Publ. Inst. Math. (Beograd) (N.S.) 42(56) (1987), 107–121.

The usual conformal transformation of the Norden metric g is defined by

$$\overline{g} = e^{2u} g,$$

where u is a pluriharmonic function, i.e. the 1-form $du \circ J$ is closed.

A $\mathcal{W}_1$-manifold with closed Lie 1-forms θ and θ^* is called *a conformal Kähler manifold with Norden metric.*

In [3] it is proved that such a manifold is conformally equivalent to a Kähler manifold with Norden metric by the usual conformal transformation. The subclass of these manifolds is denoted by $\mathcal{W}_1^0$.

It is known that on a pseudo-Riemannian manifold M ($\dim M = 2n \geq 4$) the conformal invariant *Weyl tensor* has the form

$$W(R) = R - \frac{1}{2(n-1)}\{\psi_1(\rho) - \frac{\tau}{2n-1}\pi_1\}.$$

Let L be a Kähler curvature-like tensor on (M, J, g), $\dim M = 2n \geq 6$. Then the *Bochner tensor* $B(L)$ for L is defined by [3]:

$$B(L) = L - \tfrac{1}{2(n-2)}\{\psi_1 - \psi_2\}\big(\rho(L)\big) \\ + \tfrac{1}{4(n-1)(n-2)}\{\tau(L)\big(\pi_1 - \pi_2\big) + \tau^*(L)\pi_3\}.$$

In [3] it is proved that the Bochner tensor of the canonical connection is a conformal invariant of the canonical conformal group generated by the usual conformal transformation on a $\mathcal{W}_1^0$-manifold.

2. Complex Connections on $\mathcal{W}_1$-manifolds

Definition 2.1. [4]* A linear connection ∇' on an almost complex manifold (M, J) is said to be *almost complex* if $\nabla' J = 0$.

Theorem 2.1. *On a $\mathcal{W}_1$-manifold with Norden metric there exists an 8-parametric family of complex connections ∇' defined by*

$$\nabla'_x y = \nabla_x y + Q(x, y) \tag{2.1}$$

where the deformation tensor $Q(x, y)$ is given by

$$\begin{aligned} Q(x,y) &= \frac{1}{2n}\left[\theta(Jy)x - g(x,y)J\Omega\right] \\ &+\frac{1}{n}\{\lambda_1\theta(x)y + \lambda_2\theta(x)Jy + \lambda_3\theta(Jx)y + \lambda_4\theta(Jx)Jy \\ &\quad +\lambda_5\left[\theta(y)x - \theta(Jy)Jx\right] + \lambda_6\left[\theta(y)Jx + \theta(Jy)x\right] \\ &\quad +\lambda_7\left[g(x,y)\Omega - g(x,Jy)J\Omega\right] + \lambda_8\left[g(x,Jy)\Omega + g(x,y)J\Omega\right]\}, \end{aligned} \tag{2.2}$$

$\lambda_i \in \mathbb{R}$, $i = 1, 2, ..., 8$.

*[4] S. Kobayshi, K. Nomizu, *Foundations of differential geometry* vol. 1, 2, Intersc. Publ., New York, 1963, 1969.

Remark 2.1. The 2-parametric family of complex connections obtained for

$$\lambda_1 = \lambda_4, \qquad \lambda_3 = -\lambda_2, \qquad \lambda_5 = \lambda_7 = 0, \qquad \lambda_8 = -\lambda_6 = \frac{1}{4}$$

is studied by us in [7].

Theorem 2.2. *The complex connections ∇' defined by (2.1) and (2.2) are symmetric on a $\mathcal{W}_1$-manifold if and only if*

$$\lambda_1 = -\lambda_4 = \lambda_5, \qquad \lambda_2 = \lambda_3 - \frac{1}{2} = \lambda_6.$$

Then, by putting $\lambda_1 = -\lambda_4 = \lambda_5 = \mu_1$, $\lambda_2 = \lambda_6 = \lambda_3 - \frac{1}{2} = \mu_2$, $\lambda_7 = \mu_3$, $\lambda_8 = \mu_4$ in (2.2), we obtain a 4-parametric family of complex symmetric connections ∇'' on a $\mathcal{W}_1$-manifold which are defined by

$$\begin{aligned} \nabla''_x y = \nabla_x y &+ \tfrac{1}{2n}\left[\theta(Jx)y + \theta(Jy)x - g(x,y)J\Omega\right] \\ &+ \tfrac{1}{n}\left\{\boldsymbol{\mu_1}\left[\theta(x)y + \theta(y)x - \theta(Jx)Jy - \theta(Jy)Jx\right]\right. \\ &+ \boldsymbol{\mu_2}\left[\theta(Jx)y + \theta(Jy)x + \theta(x)Jy + \theta(y)Jx\right] \\ &+ \left.\boldsymbol{\mu_3}\left[g(x,y)\Omega - g(x,Jy)J\Omega\right] + \boldsymbol{\mu_4}\left[g(x,Jy)\Omega + g(x,y)J\Omega\right]\right\}. \end{aligned} \tag{2.3}$$

The well-known Yano connection [8,9]* on a $\mathcal{W}_1$-manifold with Norden metric is obtained from (2.3) for $\mu_1 = \mu_3 = 0$, $\mu_4 = -\mu_2 = \frac{1}{4}$.

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*[8] K. Yano, *Affine connections in an almost product space*, Kodai Math. Semin. Rep. 11(1) (1959), 1–24.
∗ [9] K. Yano, *Differential geometry on complex and almost complex spaces*, Pure and Applied Math. vol. 49, Pergamon Press Book, New York, 1965.

Definition 2.2. [2]* A linear connection ∇' on an almost complex manifold with Norden metric (M, J, g) is said to be *natural* if

$$\nabla' J = \nabla' g = 0 \quad \Longleftrightarrow \quad \nabla' g = \nabla' \widetilde{g} = 0.$$

Theorem 2.3. *The complex connections ∇' defined by (2.1) and (2.2) are natural on a $\mathcal{W}_1$-manifold if and only if*

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0, \qquad \lambda_7 = -\lambda_5, \qquad \lambda_8 = -\lambda_6.$$

*[2] G. Ganchev, V. Mihova, *Canonical connection and the canonical conformal group on an almost complex manifold with B-metric*, Ann. Univ. Sofia Fac. Math. Inform., 81(1) (1987), 195–206.

If we put $\lambda_8 = -\lambda_6 = s$, $\lambda_7 = -\lambda_5 = t$, $\lambda_i = 0$, $i = 1, 2, 3, 4$, in (2.2) we obtain a 2-parametric family of natural connections ∇''' defined by

$$\begin{aligned}\nabla'''_x y = \nabla_x y + \tfrac{1-2\mathbf{s}}{2n}\left[\theta(Jy)x - g(x,y)J\Omega\right] \\ + \tfrac{1}{n}\{\mathbf{s}\left[g(x,Jy)\Omega - \theta(y)Jx\right] \\ +\mathbf{t}\left[g(x,y)\Omega - g(x,Jy)J\Omega - \theta(y)x + \theta(Jy)Jx\right]\}.\end{aligned} \tag{2.4}$$

The well-known canonical connection [2] (or B-connection [3]*) on a $\mathcal{W}_1$-manifold with Norden metric is obtained from (2.4) for $s = \frac{1}{4}$, $t = 0$.

*[3] G. Ganchev, K. Gribachev, V. Mihova, *B-connections and their conformal invariants on conformally Kähler manifolds with B-metric*, Publ. Inst. Math. (Beograd) (N.S.) 42(56) (1987), 107–121.

We give a summery of the obtained results in the following table

Connection type	*Symbol*	*Parameters*
Complex	∇'	$\lambda_i \in \mathbb{R}$, $i = 1, 2, ..., 8.$
Complex symmetric	∇''	μ_i, $i = 1, 2, 3, 4$, $\mu_1 = \lambda_1 = -\lambda_4 = \lambda_5$, $\mu_2 = \lambda_2 = \lambda_6 = \lambda_3 - \frac{1}{2}$, $\mu_3 = \lambda_7$, $\mu_4 = \lambda_8$
Natural	∇'''	s, t, $s = \lambda_8 = -\lambda_6$, $t = \lambda_7 = -\lambda_5$, $\lambda_i = 0$, $i = 1, 2, 3, 4.$

Next, we study the natural connection ∇^0 obtained from (2.4) for $s = t = 0$, i.e.

$$\nabla^0_x y = \nabla_x y + \tfrac{1}{2n}\left[\theta(Jy)x - g(x,y)J\Omega\right]. \tag{2.5}$$

This connection is a semi-symmetric metric connection [10], [6]*.

Let $R^0(x,y,z,u) = g(R^0(x,y)z,u)$.

Proposition 2.4. *On a $\mathcal{W}_1$-manifold with closed Lie 1-form θ^* the Kähler curvature tensor R^0 of ∇^0 has the form*

$$R^0 = R - \frac{1}{2n}\psi_1(P),$$

$$P(x,y) = (\nabla_x\theta)\,Jy + \frac{1}{2n}\theta(x)\theta(y) + \frac{\theta(\Omega)}{4n}g(x,y) + \frac{\theta(J\Omega)}{2n}g(x,Jy).$$

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*[6] S. D. Singh, A. K. Pandey, *Semi-symmetric metric connections in an almost Norden metric manifold*, Acta Cienc. Indica Math. 27(1) (2001), 43–54.

*[10] K. Yano, *On semi-symmetric metric connection*, Rev. Roumanie Math. Pure Appl. 15 (1970), 1579–1586.

Proposition 2.5. *Let (M, J, g) be a $\mathcal{W}_1$-manifold with closed Lie 1-form θ^*, and ∇^0 be the natural connection defined by (2.5). Then, the Weyl tensor is invariant by the transformation $\nabla \to \nabla^0$, i.e.*

$$W(R^0) = W(R).$$

Remark 2.2. The above statement is a well-known fact for a semi-symmetric metric connection.

Let $R'(x, y, z, u) = g(R'(x, y)z, u)$ be the curvature tensor of ∇', $\lambda_i \in \mathbb{R}$, $i = 1, 2, ..., 8$. Then, R' is a Kähler tensor on a conformal Kähler manifold with Norden metric iff

$$\lambda_7 = -\lambda_5, \qquad \lambda_8 = -\lambda_6.$$

Theorem 2.6. *Let (M, J, g) be a conformal Kähler manifold with Norden metric, and ∇' be the complex connections defined by (2.1) and (2.2). Then R' is a Kähler curvature tensor on M if and only if $\lambda_7 = -\lambda_5$ and $\lambda_8 = -\lambda_6$. In this case from (2.1) and (2.2) we obtain a 6-parametric family of complex connections ∇' whose curvature tensors R' have the form*

$$R' = R^0 + \frac{\lambda_7}{n}\{\psi_1 - \psi_2\}(S_1) + \frac{\lambda_8}{n}\{\psi_1 - \psi_2\}(S_2)$$

$$+ \frac{\lambda_7(4\lambda_8 - 1)}{2n^2}\{\psi_1 - \psi_2\}(S_3) + \frac{\lambda_7(1 - 2\lambda_8)\theta(J\Omega)}{n^2}\{\pi_1 - \pi_2\}$$

$$+ \frac{2\lambda_7\lambda_8\theta(\Omega)}{n^2}\pi_3,$$

where R^0 is the curvature tensor of ∇^0 defined by (2.5) and

$$S_1(x,y) = (\nabla_x \theta)\, y + \frac{\lambda_7}{n}[\theta(x)\theta(y) - \theta(Jx)\theta(Jy)] - \frac{\lambda_7 \theta(\Omega)}{2n} g(x,y)$$

$$+ \frac{\lambda_7 \theta(J\Omega)}{2n} g(x, Jy),$$

$$S_2(x,y) = (\nabla_x \theta)\, Jy + \frac{1-2\lambda_8}{2n}[\theta(x)\theta(y) - \theta(Jx)\theta(Jy)]$$

$$+ \frac{\lambda_8 \theta(\Omega)}{2n} g(x,y) + \frac{(1-\lambda_8)\theta(J\Omega)}{2n} g(x, Jy),$$

$$S_3(x,y) = \theta(x)\theta(Jy) + \theta(Jx)\theta(y).$$

Corollary 2.1. *Let* (M, J, g) *be a conformal Kähler manifold with Norden metric and* ∇' *be the 8-parametric family of complex connections defined by (2.1) and (2.2). Then,*

$$R' = R^0$$

if and only if $\lambda_i = 0$ *for* $i = 5, 6, 7, 8$.

Corollary 2.2. *On a conformal Kähler manifold with Norden metric the Weyl tensor is invariant by the transformation of the Levi-Civita connection in any of the complex connection* ∇' *defined by (2.1) and (2.2) for* $\lambda_i = 0,\ i = 5, 6, 7, 8$.

Theorem 2.7. *Let* (M, J, g) *be a conformal Kähler manifold with Norden metric,* R' *be the curvature tensor of* ∇' *defined by (2.1) and (2.2) for* $\lambda_7 = -\lambda_5$, $\lambda_8 = -\lambda_6$ *and* R^0 *be the curvature tensor of* ∇^0 *given by (2.5). Then the Bochner tensor is invariant by the transformations* $\nabla^0 \to \nabla'$, *i.e.*

$$B(R') = B(R^0).$$

3. Conformal Transformations of Complex Connections

Let (M, J, g) and $(M, J, \bar{g})$ be conformally equivalent almost complex manifolds with Norden metric by the transformation $\bar{g} = e^{2u}g$. It is known that the Levi-Civita connections ∇ and $\overline{\nabla}$ of g and $\bar{g}$, respectively, are related as follows

$$\overline{\nabla}_x y = \nabla_x y + \sigma(x)y + \sigma(y)x - g(x,y)\Theta,$$

$\sigma(x) = du(x)$ and $\Theta = \mathrm{grad}\sigma$, i.e. $\sigma(x) = g(x, \Theta)$.

Lemma 3.1. *Let (M, J, g) be an almost complex manifold with Norden metric and $(M, J, \bar{g})$ be its conformally equivalent manifold by the transformation $\bar{g} = e^{2u}g$. Then the curvature tensors R and $\bar{R}$ of ∇ and $\overline{\nabla}$, respectively, are related as follows*

$$\bar{R} = e^{2u}\{R - \psi_1(V) - \pi_1\sigma(\Theta)\},$$

where $V(x,y) = \big(\nabla_x\sigma\big)y - \sigma(x)\sigma(y)$.

Let us first study the conformal group of the natural connection ∇^0 given by (2.5).

$$\overline{\nabla}^0_x y = \nabla^0_x y + \sigma(x) y.$$

Theorem 3.1. *Let (M, J, g) be a $\mathcal{W}_1$-manifold with closed Lie 1-form θ^*. Then the curvature tensor R^0 of ∇^0 is conformally invariant, i.e.*

$$\bar{R}^0 = e^{2u} R^0.$$

Theorem 3.3. *On a conformal Kähler manifold with Norden metric the Bochner curvature tensor of the complex connections* ∇' *defined by (2.1) and (2.2) with the conditions* $\lambda_7 = -\lambda_5$ *and* $\lambda_8 = -\lambda_6$ *is conformally invariant by the transformation* $\bar{g} = e^{2u}g$, *i.e.*

$$B(\bar{R}') = e^{2u}B(R').$$

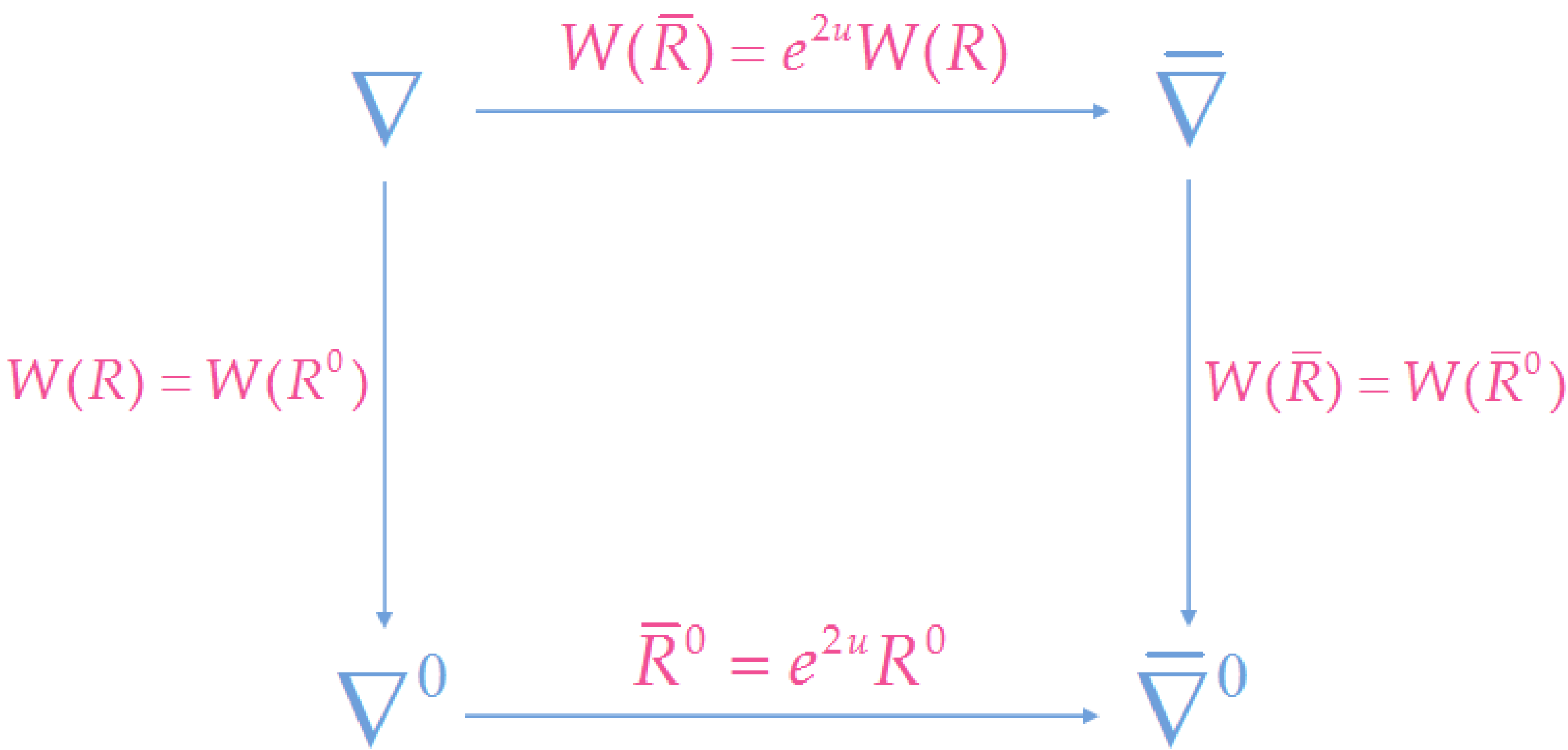
Remark 3.1. G. Ganchev, K. Gribachev, V. Mihova have proved in [3] such statement for the canonical connection.

Corollary 3.1. *Let* (M, J, g) *be a conformal Kähler manifold with Norden metric and* ∇' *be a complex connection defined by (2.1) and (2.2). If* $\lambda_i = 0$ *for* $i = 5, 6, 7, 8$, *then the curvature tensor of* ∇' *is conformally invariant by the transformation* $\bar{g} = e^{2u}g$.

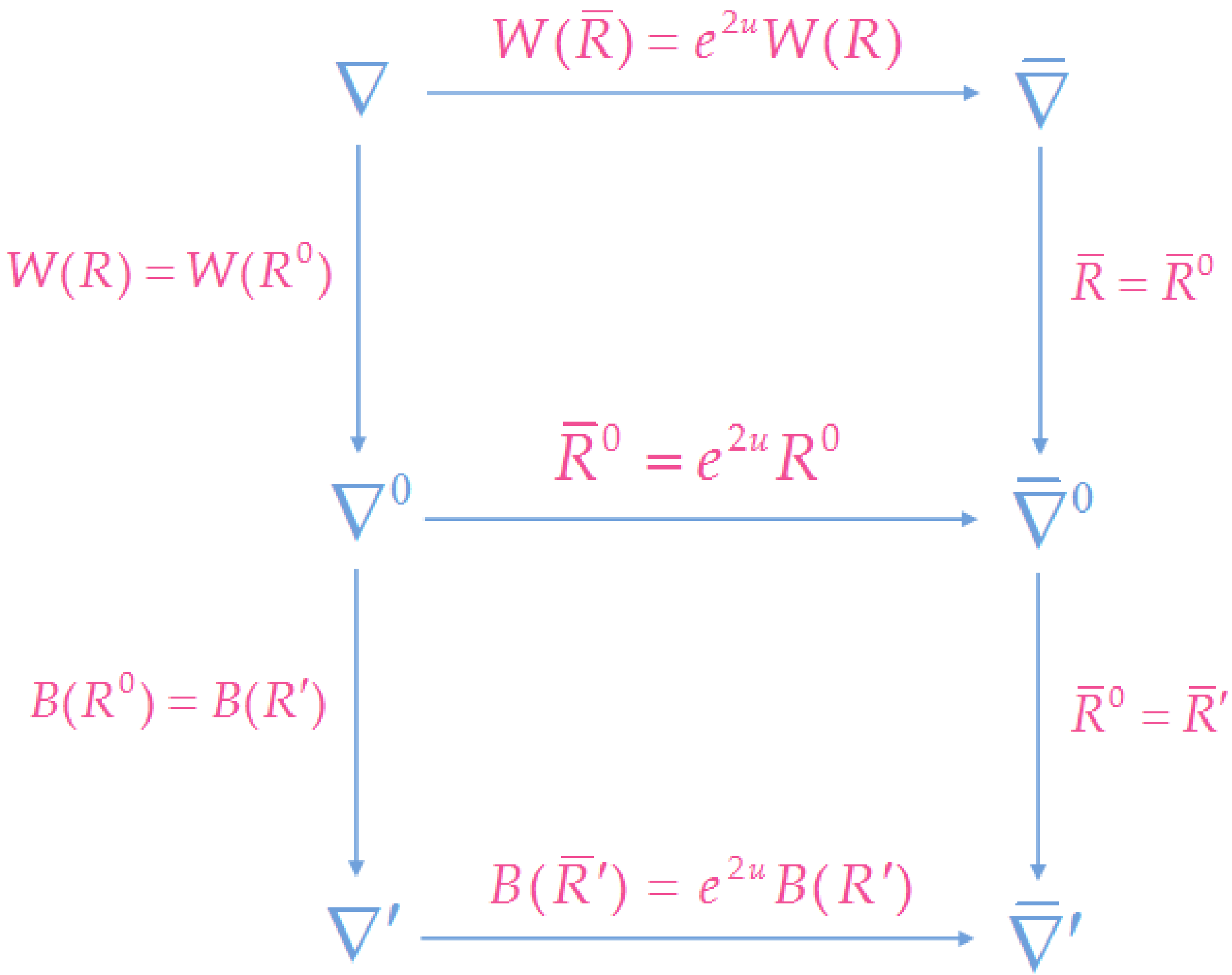
(M, J, g) – $\mathcal{W}_1$-manifold with closed 1-form $\theta \circ J$

∇ – L.C.

∇^0 – S.S.M.C. obtained from (2.1) and (2.2) for $\lambda_i = 0$, $i = 1, 2, ..., 8$.



(M, J, g) – conformal Kähler manifold with Norden metric
∇' – 6-parametric family of complex connections obtained from (2.1) and (2.2) for $\lambda_7 = -\lambda_5$, $\lambda_8 = -\lambda_6$.



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~Thank you!~