Complex Connections on Conformal Kähler Manifolds with Norden Metric

Marta Teofilova, Ph.D.
Faculty of Mathematics and Informatics
University of Plovdiv

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1. Preliminaries

Let \((M, J, g)\) be a \(2n\)-dimensional almost complex manifold with Norden metric, i.e. \(J\) is an almost complex structure and \(g\) is a pseudo-Riemannian metric on \(M\) such that

\[
J^2 x = -x, \quad g(Jx, Jy) = -g(x, y), \quad x, y \in \mathfrak{X}(M).
\]

The associated metric \(\tilde{g}\) of \(g\) is given by

\[
\tilde{g}(x, y) = g(x, Jy)
\]

and is a Norden metric, too. Both metrics are necessarily of signature \((n, n)\).
Let $\nabla$ be the Levi-Civita connection of the metric $g$. The fundamental tensor field $F$ of type $(0, 3)$ on $M$ is defined by

$$F(x, y, z) = g((\nabla_x J)y, z)$$

and has the following symmetries

$$F(x, y, z) = F(x, z, y) = F(x, Jy, Jz).$$

The Lie 1-forms $\theta$ and $\theta^*$ associated with $F$, and the Lie vector $\Omega$, corresponding to $\theta$, are defined by, respectively

$$\theta(x) = g^{ij} F(e_i, e_j, x), \quad \theta^* = \theta \circ J, \quad \theta(x) = g(x, \Omega),$$

where $\{e_i\}$ ($i = 1, 2, \ldots, 2n$) is an arbitrary basis of $T_pM$ at a point $p$ of $M$, and $g^{ij}$ are the components of the inverse matrix of $g$ with respect to the basis $\{e_i\}$. 
A classification of the almost complex manifolds with Norden metric is introduced by G. Ganchev and A. Borisov in [1]*, where eight classes of these manifolds are characterized according to the properties of \( F \). The three basic classes \( \mathcal{W}_i \) \((i = 1, 2, 3)\) are given by, respectively

- the class \( \mathcal{W}_1 \):

\[
F(x, y, z) = \frac{1}{2n} \left[ g(x, y)\theta(z) + g(x, Jy)\theta(Jz) \\
+ g(x, z)\theta(y) + g(x, Jz)\theta(Jy) \right];
\]

- the class \( \mathcal{W}_2 \) of the special complex manifolds with Norden metric:

\[
F(x, y, Jz) + F(y, z, Jx) + F(z, x, Jy) = 0, \quad \theta = 0;
\]

• the class $\mathcal{W}_3$ of the quasi-Kähler manifolds with Norden metric:

$$F(x, y, z) + F(y, z, x) + F(z, x, y) = 0.$$ 

The special class $\mathcal{W}_0$ of the Kähler manifolds with Norden metric is characterized by the condition $F = 0$ and is contained in each of the other classes.

Let $R$ be the curvature tensor of $\nabla$, i.e.

$$R(x, y)z = \nabla_x \nabla_y z - \nabla_y \nabla_x z - \nabla_{[x,y]} z.$$

The corresponding $(0,4)$-type tensor is defined by

$$R(x, y, z, u) = g(R(x, y)z, u).$$
A tensor $L$ of type (0,4) is said to be *curvature-like* if it has the properties of $R$, i.e.

\[
L(x, y, z, u) = -L(y, x, z, u) = -L(x, y, u, z),
\]

\[
L(x, y, z, u) + L(y, z, x, u) + L(z, x, y, u) = 0.
\]

The Ricci tensor $\rho(L)$ and the scalar curvatures $\tau(L)$ and $\tau^*(L)$ of $L$ are defined by:

\[
\rho(L)(y, z) = g^{ij} L(e_i, y, z, e_j),
\]

\[
\tau(L) = g^{ij} \rho(L)(e_i, e_j), \quad \tau^*(L) = g^{ij} \rho(L)(e_i, Je_j).
\]

A curvature-like tensor $L$ is called a *Kähler tensor* if

\[
L(x, y, Jz, Ju) = -L(x, y, z, u).
\]
Let $S$ be a tensor of type $(0,2)$. We consider the following tensors [3]*:

$$\psi_1(S)(x, y, z, u) = g(y, z)S(x, u) - g(x, z)S(y, u)$$
$$+ g(x, u)S(y, z) - g(y, u)S(x, z),$$

$$\psi_2(S)(x, y, z, u) = g(y, Jz)S(x, Ju) - g(x, Jz)S(y, Ju)$$
$$+ g(x, Ju)S(y, Jz) - g(y, Ju)S(x, Jz),$$

$$\pi_1 = \frac{1}{2}\psi_1(g), \quad \pi_2 = \frac{1}{2}\psi_2(g), \quad \pi_3 = -\psi_1(\tilde{g}) = \psi_2(\tilde{g}).$$

$\psi_1(S)$ is curvature-like if $S$ is symmetric;

$\psi_2(S)$ is curvature-like is $S$ is symmetric and hybrid with respect to $J$, i.e. $S(x, Jy) = S(y, Jx)$;

In the last case the tensor \{ $\psi_1 - \psi_2$ \}(S) is Kählerian.

The usual conformal transformation of the Norden metric $g$ is defined by

$$\bar{g} = e^{2u} g,$$

where $u$ is a pluriharmonic function, i.e. the 1-form $du \circ J$ is closed.

A $\mathcal{W}_1$-manifold with closed Lie 1-forms $\theta$ and $\theta^*$ is called a conformal Kähler manifold with Norden metric.

In [3] it is proved that such a manifold is conformally equivalent to a Kähler manifold with Norden metric by the usual conformal transformation. The subclass of these manifolds is denoted by $\mathcal{W}_1^0$. 
It is known that on a pseudo-Riemannian manifold $M$ (dim $M = 2n \geq 4$) the conformal invariant Weyl tensor has the form

$$W(R) = R - \frac{1}{2(n-1)}\{\psi_1(\rho) - \frac{\tau}{2n-1}\pi_1\}.$$

Let $L$ be a Kähler curvature-like tensor on $(M, J, g)$, dim $M = 2n \geq 6$. Then the Bochner tensor $B(L)$ for $L$ is defined by [3]:

$$B(L) = L - \frac{1}{2(n-2)}\{\psi_1 - \psi_2\}(\rho(L))$$

$$+ \frac{1}{4(n-1)(n-2)}\{\tau(L)(\pi_1 - \pi_2) + \tau^*(L)\pi_3\}.$$

In [3] it is proved that the Bochner tensor of the canonical connection is a conformal invariant of the canonical conformal group generated by the usual conformal transformation on a $\mathcal{W}^0_1$-manifold.
2. Complex Connections on $\mathcal{W}_1$-manifolds

**Definition 2.1.** [4]* A linear connection $\nabla'$ on an almost complex manifold $(M, J)$ is said to be almost complex if $\nabla' J = 0$.

**Theorem 2.1.** On a $\mathcal{W}_1$-manifold with Norden metric there exists an 8-parametric family of complex connections $\nabla'$ defined by

$$\nabla'_{xy} = \nabla_{xy} + Q(x, y)$$

where the deformation tensor $Q(x, y)$ is given by

$$Q(x, y) = \frac{1}{2n} \left[ \theta(Jy)x - g(x, y)J\Omega \right]$$

$$+ \frac{1}{n} \left\{ \lambda_1 \theta(x)y + \lambda_2 \theta(x)Jy + \lambda_3 \theta(Jx)y + \lambda_4 \theta(Jx)Jy 
+ \lambda_5 \left[ \theta(y)x - \theta(Jy)Jx \right] + \lambda_6 \left[ \theta(y)Jx + \theta(Jy)x \right] 
+ \lambda_7 \left[ g(x, y)\Omega - g(x, Jy)J\Omega \right] + \lambda_8 \left[ g(x, Jy)\Omega + g(x, y)J\Omega \right] \right\} ,$$

$\lambda_i \in \mathbb{R}, i = 1, 2, \ldots, 8.$

Remark 2.1. The 2-parametric family of complex connections obtained for
\[
\lambda_1 = \lambda_4, \quad \lambda_3 = -\lambda_2, \quad \lambda_5 = \lambda_7 = 0, \quad \lambda_8 = -\lambda_6 = \frac{1}{4}
\]
is studied by us in [7].

Theorem 2.2. The complex connections $\nabla'$ defined by (2.1) and (2.2) are symmetric on a $\mathcal{W}_1$-manifold if and only if
\[
\lambda_1 = -\lambda_4 = \lambda_5, \quad \lambda_2 = \lambda_3 - \frac{1}{2} = \lambda_6.
\]
Then, by putting $\lambda_1 = -\lambda_4 = \lambda_5 = \mu_1$, $\lambda_2 = \lambda_6 = \lambda_3 - \frac{1}{2} = \mu_2$, $\lambda_7 = \mu_3$, $\lambda_8 = \mu_4$ in (2.2), we obtain a 4-parametric family of complex symmetric connections $\nabla''$ on a $\mathcal{W}_1$-manifold which are defined by

$$
\nabla'' _xy = \nabla _xy + \frac{1}{2n} [\theta (Jx)y + \theta (Jy)x - g(x, y) J\Omega ] \\
+ \frac{1}{n} \{ \mu_1 [\theta (x)y + \theta (y)x - \theta (Jx)Jy - \theta (Jy)Jx] \\
+ \mu_2 [\theta (Jx)y + \theta (Jy)x + \theta (x)Jy + \theta (y)Jx] \\
+ \mu_3 [g(x, y)\Omega - g(x, Jy)J\Omega ] + \mu_4 [g(x, Jy)\Omega + g(x, y)J\Omega ] \}.
$$

(2.3)

The well-known Yano connection \[8,9\]* on a $\mathcal{W}_1$-manifold with Norden metric is obtained from (2.3) for $\mu_1 = \mu_3 = 0$, $\mu_4 = -\mu_2 = \frac{1}{4}$.

**Definition 2.2.** [2]* A linear connection $\nabla'$ on an almost complex manifold with Norden metric $(M, J, g)$ is said to be *natural* if

$$\nabla' J = \nabla' g = 0 \iff \nabla' g = \nabla' \tilde{g} = 0.$$

**Theorem 2.3.** The complex connections $\nabla'$ defined by (2.1) and (2.2) are natural on a $\mathcal{W}_1$-manifold if and only if

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0, \quad \lambda_7 = -\lambda_5, \quad \lambda_8 = -\lambda_6.$$ 

If we put $\lambda_8 = -\lambda_6 = s$, $\lambda_7 = -\lambda_5 = t$, $\lambda_i = 0$, $i = 1, 2, 3, 4$, in (2.2) we obtain a 2-parametric family of natural connections $\nabla'''$ defined by

$$\nabla'''_x y = \nabla_x y + \frac{1-2s}{2n} [\theta(Jy)x - g(x, y)J\Omega]$$

$$+ \frac{1}{n} \left\{ s \left[ g(x, Jy)\Omega - \theta(y)Jx \right] 
+ t \left[ g(x, y)\Omega - g(x, Jy)J\Omega - \theta(y)x + \theta(Jy)Jx \right] \right\}.$$  \hspace{1cm} (2.4)

The well-known canonical connection [2] (or $B$-connection [3]*) on a $W_1$-manifold with Norden metric is obtained from (2.4) for $s = \frac{1}{4}$, $t = 0$.

We give a summary of the obtained results in the following table.

<table>
<thead>
<tr>
<th>Connection type</th>
<th>Symbol</th>
<th>Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>Complex</td>
<td>$\nabla'$</td>
<td>$\lambda_i \in \mathbb{R}, i = 1, 2, \ldots, 8$.</td>
</tr>
<tr>
<td>Complex</td>
<td>$\nabla''$</td>
<td>$\mu_i, i = 1, 2, 3, 4,$</td>
</tr>
<tr>
<td>symmetric</td>
<td></td>
<td>$\mu_1 = \lambda_1 = -\lambda_4 = \lambda_5, \mu_2 = \lambda_2 = \lambda_6 = \lambda_3 - \frac{1}{2}$, $\mu_3 = \lambda_7, \mu_4 = \lambda_8$</td>
</tr>
<tr>
<td>Natural</td>
<td>$\nabla'''$</td>
<td>$s, t,$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$s = \lambda_8 = -\lambda_6, t = \lambda_7 = -\lambda_5,$ $\lambda_i = 0, i = 1, 2, 3, 4.$</td>
</tr>
</tbody>
</table>
Next, we study the natural connection $\nabla^0$ obtained from (2.4) for $s = t = 0$, i.e.

\[
\nabla^0_xy = \nabla_xy + \frac{1}{2n} [\theta(Jy)x - g(x,y)J\Omega].
\] (2.5)

This connection is a semi-symmetric metric connection [10], [6]*. Let $R^0(x, y, z, u) = g(R^0(x, y)z, u)$.

**Proposition 2.4.** On a $\mathcal{W}_1$-manifold with closed Lie 1-form $\theta^*$ the Kähler curvature tensor $R^0$ of $\nabla^0$ has the form

\[
R^0 = R - \frac{1}{2n} \psi_1(P),
\]

\[
P(x, y) = (\nabla_x\theta) Jy + \frac{1}{2n} \theta(x)\theta(y) + \frac{\theta(\Omega)}{4n} g(x, y) + \frac{\theta(J\Omega)}{2n} g(x, Jy).
\]


Proposition 2.5. Let \((M, J, g)\) be a \(\mathcal{W}_1\)-manifold with closed Lie 1-form \(\theta^*\), and \(\nabla^0\) be the natural connection defined by (2.5). Then, the Weyl tensor is invariant by the transformation \(\nabla \to \nabla^0\), i.e.

\[
W(R^0) = W(R).
\]

Remark 2.2. The above statement is a well-known fact for a semi-symmetric metric connection.

Let \(R'(x, y, z, u) = g(R'(x, y)z, u)\) be the curvature tensor of \(\nabla'\), \(\lambda_i \in \mathbb{R}, i = 1, 2, ..., 8\). Then, \(R'\) is a Kähler tensor on a conformal Kähler manifold with Norden metric iff

\[
\lambda_7 = -\lambda_5, \quad \lambda_8 = -\lambda_6.
\]
Theorem 2.6. Let \((M, J, g)\) be a conformal Kähler manifold with Norden metric, and \(\nabla'\) be the complex connections defined by (2.1) and (2.2). Then \(R'\) is a Kähler curvature tensor on \(M\) if and only if \(\lambda_7 = -\lambda_5\) and \(\lambda_8 = -\lambda_6\). In this case from (2.1) and (2.2) we obtain a 6-parametric family of complex connections \(\nabla'\) whose curvature tensors \(R'\) have the form

\[
R' = R^0 + \frac{\lambda_7}{n} \{\psi_1 - \psi_2\} (S_1) + \frac{\lambda_8}{n} \{\psi_1 - \psi_2\} (S_2) \\
+ \frac{\lambda_7(4\lambda_8 - 1)}{2n^2} \{\psi_1 - \psi_2\} (S_3) + \frac{\lambda_7(1 - 2\lambda_8)\theta(J\Omega)}{n^2} \{\pi_1 - \pi_2\} \\
+ \frac{2\lambda_7\lambda_8\theta(\Omega)}{n^2} \pi_3,
\]

where \(R^0\) is the curvature tensor of \(\nabla^0\) defined by (2.5) and
\[S_1(x, y) = (\nabla_x \theta) y + \frac{\lambda_7}{n} [\theta(x)\theta(y) - \theta(Jx)\theta(Jy)] - \frac{\lambda_7\theta(\Omega)}{2n} g(x, y)\]
\[+ \frac{\lambda_7\theta(J\Omega)}{2n} g(x, Jy),\]

\[S_2(x, y) = (\nabla_x \theta) Jy + \frac{1-2\lambda_8}{2n} [\theta(x)\theta(y) - \theta(Jx)\theta(Jy)]\]
\[+ \frac{\lambda_8\theta(\Omega)}{2n} g(x, y) + \frac{(1-\lambda_8)\theta(J\Omega)}{2n} g(x, Jy),\]

\[S_3(x, y) = \theta(x)\theta(Jy) + \theta(Jx)\theta(y).\]
Corollary 2.1. Let $(M, J, g)$ be a conformal Kähler manifold with Norden metric and $\nabla'$ be the 8-parametric family of complex connections defined by (2.1) and (2.2). Then,

$$R' = R^0$$

if and only if $\lambda_i = 0$ for $i = 5, 6, 7, 8$.

Corollary 2.2. On a conformal Kähler manifold with Norden metric the Weyl tensor is invariant by the transformation of the Levi-Civita connection in any of the complex connection $\nabla'$ defined by (2.1) and (2.2) for $\lambda_i = 0$, $i = 5, 6, 7, 8$. 
Theorem 2.7. Let \((M, J, g)\) be a conformal Kähler manifold with Norden metric, \(R'\) be the curvature tensor of \(\nabla'\) defined by (2.1) and (2.2) for \(\lambda_7 = -\lambda_5, \lambda_8 = -\lambda_6\) and \(R^0\) be the curvature tensor of \(\nabla^0\) given by (2.5). Then the Bochner tensor is invariant by the transformations \(\nabla^0 \to \nabla'\), i.e.

\[ B(R') = B(R^0). \]
3. Conformal Transformations of Complex Connections

Let $(M, J, g)$ and $(M, J, \bar{g})$ be conformally equivalent almost complex manifolds with Norden metric by the transformation $\bar{g} = e^{2u}g$. It is known that the Levi-Civita connections $\nabla$ and $\bar{\nabla}$ of $g$ and $\bar{g}$, respectively, are related as follows

$$\bar{\nabla}xy = \nabla xy + \sigma(x)y + \sigma(y)x - g(x, y)\Theta,$$

where $\sigma(x) = du(x)$ and $\Theta = \text{grad} \sigma$, i.e. $\sigma(x) = g(x, \Theta)$.

**Lemma 3.1.** Let $(M, J, g)$ be an almost complex manifold with Norden metric and $(M, J, \bar{g})$ be its conformally equivalent manifold by the transformation $\bar{g} = e^{2u}g$. Then the curvature tensors $R$ and $\bar{R}$ of $\nabla$ and $\bar{\nabla}$, respectively, are related as follows

$$\bar{R} = e^{2u} \left\{ R - \psi_1(V) - \pi_1 \sigma(\Theta) \right\},$$

where $V(x, y) = (\nabla_x \sigma)y - \sigma(x)\sigma(y)$. 
Let us first study the conformal group of the natural connection $\nabla^0$ given by (2.5).

$$\nabla^0_{xy} = \nabla^0_{xy} + \sigma(x)y.$$ 

**Theorem 3.1.** Let $(M, J, g)$ be a $\mathcal{W}_1$-manifold with closed Lie 1-form $\theta^*$. Then the curvature tensor $R^0$ of $\nabla^0$ is conformally invariant, i.e.

$$\bar{R}^0 = e^{2u} R^0.$$
**Theorem 3.3.** On a conformal Kähler manifold with Norden metric the Bochner curvature tensor of the complex connections $\nabla'$ defined by (2.1) and (2.2) with the conditions $\lambda_7 = -\lambda_5$ and $\lambda_8 = -\lambda_6$ is conformally invariant by the transformation $\bar{g} = e^{2u}g$, i.e.

$$B(\bar{R}') = e^{2u} B(R').$$

**Remark 3.1.** G. Ganchev, K. Gribachev, V. Mihova have proved in [3] such statement for the canonical connection.

**Corollary 3.1.** Let $(M, J, g)$ be a conformal Kähler manifold with Norden metric and $\nabla'$ be a complex connection defined by (2.1) and (2.2). If $\lambda_i = 0$ for $i = 5, 6, 7, 8$, then the curvature tensor of $\nabla'$ is conformally invariant by the transformation $\bar{g} = e^{2u}g$. 
\((M, J, g) - \mathcal{W}_1\)-manifold with closed 1-form \(\theta \circ J\)

\(\nabla - \text{L.C.}\)

\(\nabla^0 - \text{S.S.M.C. obtained from (2.1) and (2.2) for } \lambda_i = 0, i = 1, 2, \ldots, 8.\)
\((M, J, g)\) – conformal Kähler manifold with Norden metric

\(\nabla'\) – 6-parametric family of complex connections obtained from (2.1) and (2.2) for \(\lambda_7 = -\lambda_5, \lambda_8 = -\lambda_6\).


~Thank you!~