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Almost Complex Connections on Almost Complex Manifolds with Norden Metric

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9th International Workshop on Complex Structures, Integrability and Vector Fields
Sofia 2008

1 ALMOST COMPLEX MANIFOLDS WITH NORDEN METRIC

Let (M, J, g) be a $2n$-dimensional **almost complex manifold with Norden metric**, i.e. J is an almost complex structure and g is a Norden metric on M such that:

$$J^2 = -\mathrm{id}, \qquad g(JX, JY) = -g(X, Y), \qquad X, Y \in \mathfrak{X}(M). \tag{1.1}$$

The associated metric $\widetilde{g}$ is defined by

$$\widetilde{g}(X, Y) = g(X, JY) \tag{1.2}$$

and it is a Norden metric, too. Both metrics are indefinite of signature (n, n).

Let ∇ be the Levi-Civita connection of g. **The tensor field** F of type (0,3) is given by

$$F(X,Y,Z) = g\left((\nabla_X J)Y, Z\right) \tag{1.3}$$

and has the properties:

$$F(X,Y,Z) = F(X,Z,Y), \qquad F(X,JY,JZ) = F(X,Y,Z). \tag{1.4}$$

The Lie 1-forms θ and θ^* associated with F and **the Lie vector** Ω corresponding to θ are defined by:

$$\theta(x) = g^{ij}F(e_i,e_j,x), \qquad \theta^* = \theta \circ J, \qquad g(x,\Omega) = \theta(x), \tag{1.5}$$

where $\{e_i\}$ $(i = 1,2,\dots,2n)$ is an arbitrary base of the tangent space T_pM, $p \in M$, and g^{ij} are the components of the reverse matrix of the matrix (g_{ij}).

The **Nijenhuis tensor field** N for J is given by

$$N(X,Y) = [JX, JY] - [X,Y] - J[JX,Y] - J[X,JY]. \tag{1.6}$$

$$N(X,Y) = (\nabla_X J)JY - (\nabla_Y J)JX + (\nabla_{JX} J)Y - (\nabla_{JY} J)X. \tag{1.7}$$

$$N(X,Y,Z) = g(N(X,Y),Z) \tag{1.8}$$

It is well known [7]* that the almost complex structure is complex if and only if it is integrable, i.e. $N = 0$.

The associated tensor $\widetilde{N}$ of the Nijenhuis tensor N is defined by [2]*:

$$\widetilde{N}(X,Y) = (\nabla_X J)JY + (\nabla_Y J)JX + (\nabla_{JX} J)Y + (\nabla_{JY} J)X. \tag{1.9}$$

$$\widetilde{N}(X,Y,Z) = g(\widetilde{N}(X,Y),Z) \tag{1.10}$$

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*[7] **A. Newlander, L. Niremberg**, *Complex analytic coordinates in almost complex manifolds*, Ann. Math. **65**, 1957, 391–404.

*[2] **G. Ganchev, A. Borisov**, *Note on the almost complex manifolds with a Norden metric*, Compt. Rend. Acad. Bulg. Sci. **39**(5), 1986, 31–34.

A classification of the almost complex manifolds with Norden metric is introduced in [2]*, where eight classes of the considered manifolds are characterized according to the properties of F.

- **The class $\mathcal{W}_1$**

$$F(X,Y,Z)=\frac{1}{2n}\big\{g(X,Y)\theta(Z)+g(X,Z)\theta(Y)+g(X,JY)\theta(JZ)+g(X,JZ)\theta(JY)\big\};\tag{1.11}$$

- The class $\mathcal{W}_2$ of the **special complex manifolds with Norden metric**

$$F(X,Y,JZ)+F(Y,Z,JX)+F(Z,X,JY)=0,\quad \theta=0;\tag{1.12}$$

- The class $\mathcal{W}_3$ of the **quasi-Kähler manifolds with Norden metric**

$$F(X,Y,Z)+F(Y,Z,X)+F(Z,X,Y)=0\quad\Leftrightarrow\quad \widetilde{N}=0;\tag{1.13}$$

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*[2] **G. Ganchev, A. Borisov**, *Note on the almost complex manifolds with a Norden metric*, Compt. Rend. Acad. Bulg. Sci. **39**(5), 1986, 31–34.

- The class $\mathcal{W}_1 \oplus \mathcal{W}_2$ of the **complex manifolds with Norden metric**

$$F(X,Y,JZ) + F(Y,Z,JX) + F(Z,X,JY) = 0 \Leftrightarrow N = 0. \tag{1.14}$$

The special class $\mathcal{W}_0$ of **the Kähler manifolds with Norden metric** is characterized by the condition $F = 0$ and it is contained in each of the other classes.

The curvature tensor R of ∇ is defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \quad R(X,Y,Z,W) = g(R(X,Y)Z,W).$$

A tensor L of type (0,4) is called ***curvature-like*** if it has the properties of R, i.e.

$$L(X,Y,Z,W) = -L(Y,X,Z,W) = -L(X,Y,W,Z),$$
$$L(X,Y,Z,W) + L(Y,Z,X,W) + L(Z,X,Y,W) = 0.$$

Then, the Ricci tensor $\rho(L)$ and the scalar curvatures $\tau(L)$ and $\tau^*(L)$ of a curvature-like tensor L are given by:

$$\rho(L)(X,Y) = g^{ij} L(e_i, X, Y, e_j),$$
$$\tau(L) = g^{ij} \rho(L)(e_i, e_j), \qquad \tau^*(L) = g^{ij} \rho(L)(e_i, Je_j).$$

A curvature-like tensor L is called ***a Kähler tensor*** if

$$L(X,Y,JZ,JW) = -L(X,Y,Z,W).$$

Let S be a tensor of type (0,2). The essential curvature-like tensors on an almost complex manifold with Norden metric are:

$$\begin{aligned}\psi_1(S)(X,Y,Z,W) = {} & g(Y,Z)S(X,W) - g(X,Z)S(Y,W) \\ & + g(X,W)S(Y,Z) - g(Y,W)S(X,Z),\end{aligned}$$

$$\begin{aligned}\psi_2(S)(X,Y,Z,W) = {} & g(Y,JZ)S(X,JW) - g(X,JZ)S(Y,JW) \\ & + g(X,JW)S(Y,JZ) - g(Y,JW)S(X,JZ),\end{aligned}$$

$$\pi_1 = \tfrac{1}{2}\psi_1(g), \qquad \pi_2 = \tfrac{1}{2}\psi_2(g), \qquad \pi_3 = -\psi_1(\widetilde{g}) = \psi_2(\widetilde{g}).$$

The tensors $\pi_1 - \pi_2$ and π_3 are Kählerian.

2 ALMOST COMPLEX CONNECTIONS ON ALMOST COMPLEX MANIFOLDS WITH NORDEN METRIC

Definition 2.1 [6]* A linear connection ∇' on an almost complex manifold (M, J) is called ***almost complex*** if $\nabla' J = 0$.

Theorem 2.1 *On an almost complex manifold with Norden metric there exists a 4-parametric family of almost complex connections* ∇' *with torsion tensor* T *defined by, respectively:*

$$\begin{aligned} g\big(\nabla'_X Y - \nabla_X Y, Z\big) &= \tfrac{1}{2}F(X, JY, Z) + t_1\big\{F(Y, X, Z) + F(JY, JX, Z)\big\} \\ &+t_2\big\{F(Y, JX, Z) - F(JY, X, Z)\big\} + t_3\big\{F(Z, X, Y) + F(JZ, JX, Y)\big\} \\ &+t_4\big\{F(Z, JX, Y) - F(JZ, X, Y)\big\}, \end{aligned} \tag{2.1}$$

$$\begin{aligned} T(X, Y, Z) &= t_1\big\{F(Y, X, Z) - F(X, Y, Z) + F(JY, JX, Z) - F(JX, JY, Z)\big\} \\ &+\big(\tfrac{1}{2} - t_2\big)\big\{F(X, JY, Z) - F(Y, JX, Z)\big\} + t_2\big\{F(JX, Y, Z) - F(JY, X, Z)\big\} \\ &+2t_3 F(JZ, JX, Y) + 2t_4 F(Z, JX, Y), \end{aligned}$$

where $t_i \in \mathbb{R}$, $i = 1, 2, 3, 4$.

*[**6**] **S. Kobayshi, K. Nomizu**, *Foundations of differential geometry* vol. 1, 2, Intersc. Publ., New York, 1963, 1969.

Corollary 2.1 *On a complex manifold with Norden metric there exists a 2-parametric family of complex connections* ∇' *defined by*

$$\begin{aligned}\nabla'_X Y = \nabla_X Y + \tfrac{1}{2}(\nabla_X J)JY + p\big\{(\nabla_Y J)X + (\nabla_{JY} J)JX\big\} \\ +q\big\{(\nabla_Y J)JX - (\nabla_{JY} J)X\big\},\end{aligned} \tag{2.2}$$

where $p = t_1 + t_3$, $q = t_2 + t_4$.

Corollary 2.2 *On a quasi-Kähler manifold with Norden metric there exists a 2-parametric family of almost complex connections* ∇' *defined by*

$$\begin{aligned}\nabla'_X Y = \nabla_X Y + \tfrac{1}{2}(\nabla_X J)JY + s\big\{(\nabla_Y J)X + (\nabla_{JY} J)JX\big\} \\ +t\big\{(\nabla_Y J)JX - (\nabla_{JY} J)X\big\},\end{aligned} \tag{2.3}$$

where $s = t_1 - t_3$, $t = t_2 - t_4$.

Definition 2.2 [4]* A linear connection ∇' on an almost complex manifold with Norden metric (M, J, g) is said to be ***natural***, if

$$\nabla' J = \nabla' g = 0 \quad \Leftrightarrow \quad \nabla' g = \nabla' \widetilde{g} = 0.$$

Lemma 2.1 *Let (M, J, g) be an almost complex manifold with Norden metric and let ∇' be an arbitrary almost complex connection defined by (2.1). Then*

$$\begin{aligned} \big(\nabla'_X g\big)(Y, Z) &= (t_2 + t_4)\widetilde{N}(Y, Z, X) - (t_1 + t_3)\widetilde{N}(Y, Z, JX), \\ \big(\nabla'_X \widetilde{g}\big)(Y, Z) &= -(t_1 + t_3)\widetilde{N}(Y, Z, X) - (t_2 + t_4)\widetilde{N}(Y, Z, JX). \end{aligned} \tag{2.4}$$

Theorem 2.2 *An almost complex connection ∇' defined by (2.1) is natural on an almost complex manifold with Norden metric if and only if $t_1 = -t_3$ and $t_2 = -t_4$, i.e.*

$$g\big(\nabla'_X Y - \nabla_X Y, Z\big) = \tfrac{1}{2}F(X, JY, Z) + t_1 N(Y, Z, JX) - t_2 N(Y, Z, X). \tag{2.5}$$

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*[4] **G. Ganchev, V. Mihova**, *Canonical connection and the canonical conformal group on an almost complex manifold with B-metric*, Ann. Univ. Sofia Fac. Math. Inform. **81**(1), 1987, 195–206.

The natural connection on $\mathcal{W}_1 \oplus \mathcal{W}_2$:

$$\nabla'_X Y = \nabla_X Y + \frac{1}{2}(\nabla_X J)JY. \tag{2.6}$$

Corollary 2.3 *Let (M, J, g) be a quasi-Kähler manifold with Norden metric. Then, the connection ∇' defined by (2.3) is natural on M for all $s, t \in \mathbb{R}$.*

Definition 2.3 [4]* A natural connection ∇' with torsion tensor T on an almost complex manifold with Norden metric is said to be ***canonical*** if

$$T(X,Y,Z)+T(Y,Z,X)-T(JX,Y,JZ)-T(Y,JZ,JX)=0. \tag{2.7}$$

Proposition 2.1 *Let (M,J,g) be an almost complex manifold with Norden metric. A natural connection ∇' defined by (2.5) is canonical if and only if*

$$t_1=0, \qquad t_2=\tfrac{1}{8}.$$

In this case (2.5) takes the form

$$2g\big(\nabla'_X Y-\nabla_X Y,Z\big)=F(X,JY,Z)-\tfrac{1}{4}N(Y,Z,X).$$

Remark 2.1 G. Ganchev and V. Mihova [4] have proven that on an almost complex manifold with Norden metric there exist a unique canonical connection.

Theorem 2.3 *Let (M, J, g) be an almost complex manifold with Norden metric and non-integrable almost complex structure. Then, on M there exists a unique almost complex connection ∇' in the family (2.1) whose torsion tensor is totally skew symmetric (i.e. a 3-form). This connection is defined by*

$$g\big(\nabla'_X Y - \nabla_X Y, Z\big) = \tfrac{1}{4}\big\{2F(X, JY, Z) + F(Z, JX, Y) - F(JZ, X, Y)\big\}.$$

Corollary 2.4 *On a quasi-Kähler manifold with Norden metric there exists a unique connection ∇' in the family of natural connections (2.3) whose torsion tensor is a 3-form. This connection is given by*

$$\nabla'_X Y = \nabla_X Y + \tfrac{1}{4}\big\{2(\nabla_X J)JY - (\nabla_Y J)JX + (\nabla_{JY} J)X\big\}. \qquad (2.8)$$

Remark 2.2 The connection (2.8) can be considered as an analogue of **the Bismut connection** [1], [5]* in the geometry of the almost complex manifolds with Norden metric.

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*[**1**] **J.-M. Bismut**, *A local index theorem for non-Kähler manifolds*, Math. Ann. **284**, 1989, 681–699.
*[**5**] **P. Gauduchon**, *Hermitian connections and Dirac operators*, Bollettino U.M.I. **11**, 1997, 257–288.

Theorem 2.4 *Let (M, J, g) be a complex manifold with Norden metric. Then, on M there exists a unique complex symmetric connection ∇' belonging to the family (2.2). This connection is defined by*

$$\nabla'_X Y = \nabla_X Y + \frac{1}{4}\big\{(\nabla_X J)JY + 2(\nabla_Y J)JX - (\nabla_{JX} J)Y\big\}. \tag{2.9}$$

Remark 2.3 The connection (2.9) is known as **the Yano connection** [9], [10]*. In [8]* we have studied this connection on a complex manifold with Norden metric belonging to the class $\mathcal{W}_1$ with closed Lie forms θ and $\theta^* = \theta \circ J$, i.e. *a conformal Kähler manifold with Norden metric* and we have obtained the form of its curvature tensor.

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*[**8**] **M. Teofilova**, *Complex connections on complex manifolds with Norden metric*, In: Contemporary Aspects of Complex Analysis, Differential Geometry and Mathematical Physics, eds. S. Dimiev and K. Sekigawa, World Sci. Publ., Singapore, 2005, 326–335.

*[**9**] **K. Yano**, *Affine connections in an almost product space*, Kodai Math. Semin. Rep. **11**(1), 1959, 1–24.

*[**10**] **K. Yano**, *Differential geometry on complex and almost complex spaces*, Pure and Applied Math. vol. 49, Pergamon Press Book, New York, 1965.

A summary of the results obtained for the family of almost complex connections ∇' defined by (2.1):

	Class manifolds		
Connection	$\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$	$\mathcal{W}_1 \oplus \mathcal{W}_2$	$\mathcal{W}_3$
almost complex	$t_1, t_2, t_3, t_4 \in \mathbb{R}$	$p, q \in \mathbb{R}$	$s, t \in \mathbb{R}$
natural	$t_1 = -t_3,\ t_2 = -t_4$	$p = q = 0$	$s, t \in \mathbb{R}$
canonical	$t_1 = t_3 = 0,\ t_2 = -t_4 = \frac{1}{8}$	$p = q = 0$	$s = 0,\ t = \frac{1}{4}$
T is a 3-form	$t_1 = t_2 = t_3 = 0,\ t_4 = \frac{1}{4}$	$\nexists$	$s = 0,\ t = -\frac{1}{4}$
symmetric	$\nexists$	$p = 0,\ q = \frac{1}{4}$	$\nexists$

Table 1

$$
\begin{aligned}
\nabla'_X Y = \nabla_X Y &+ \tfrac{1}{4n}\big\{g(X,JY)\Omega - g(X,Y)J\Omega + \theta(JY)X - \theta(Y)JX\big\} \\
&+ \tfrac{p}{n}\big\{\theta(X)Y + \theta(JX)JY\big\} + \tfrac{q}{n}\big\{\theta(JX)Y - \theta(X)JY\big\}.
\end{aligned} \tag{3.1}
$$

Theorem 3.1 *Let (M, J, g) be a conformal Kähler manifold with Norden metric and ∇' be a complex connection defined by (2.2). Then, the Kähler curvature tensor R' of ∇' has the form*

$$
R' = R - \frac{1}{4n}\big\{\psi_1 + \psi_2\big\}(S) - \frac{1}{8n^2}\psi_1(P) - \frac{\theta(\Omega)}{16n^2}\big\{3\pi_1 + \pi_2\big\} + \frac{\theta(J\Omega)}{16n^2}\pi_3,
$$

where S and P are defined by, respectively:

$$
\begin{aligned}
S(X,Y) &= \big(\nabla_X \theta\big)JY + \tfrac{1}{4n}\big\{\theta(X)\theta(Y) - \theta(JX)\theta(JY)\big\}, \\
P(X,Y) &= \theta(X)\theta(Y) + \theta(JX)\theta(JY).
\end{aligned} \tag{3.2}
$$

$$\rho'(X,Y) = \rho(X,Y) - \tfrac{1}{4n}\big\{[\mathrm{div}(J\Omega) + \tfrac{\theta(\Omega)}{2}]g(X,Y) - [\mathrm{div}\Omega - \tfrac{\theta(J\Omega)}{2}]g(X,JY)$$
$$+2nS(X,Y) + \tfrac{n-1}{n}P(X,Y)\big\},$$

$$\tau' = \tau - \mathrm{div}(J\Omega) + \frac{n-1}{4n}\theta(\Omega), \qquad \tau'^* = \tau^* + \frac{n-1}{4n}\theta(J\Omega).$$

Theorem 3.2 *Let (M, J, g) be a conformal Kähler manifold with Norden metric, and τ' and τ'^* be the scalar curvatures of the Kähler tensor R' corresponding to the complex connection ∇' defined by (2.2). Then, the function $\tau' + i\tau'^*$ is holomorphic on M and the Lie 1-forms θ and θ^* are defined in a unique way by τ' and τ'^* as follows*:

$$\theta = 2nd\big(\mathrm{arctg}\frac{\tau'}{\tau'^*}\big), \qquad \theta^* = -2\mathrm{nd}\big(\ln\sqrt{\tau'^2 + \tau'^{*2}}\big).$$

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[**3**] **G. Ganchev, K. Gribachev, V. Mihova**, *B-connections and their conformal invariants on conformally Kähler manifolds with B-metric*, Publ. Inst. Math. (Beograd) (N.S.) **42**(56), 1987, 107–121.

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