Almost Complex Connections on Almost Complex Manifolds with Norden Metric

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1 ALMOST COMPLEX MANIFOLDS WITH NORDEN METRIC

Let \((M, J, g)\) be a \(2n\)-dimensional almost complex manifold with Norden metric, i.e. \(J\) is an almost complex structure and \(g\) is a Norden metric on \(M\) such that:

\[
J^2 = -\text{id}, \quad g(JX, JY) = -g(X, Y), \quad X, Y \in \mathfrak{X}(M).
\] (1.1)

The associated metric \(\tilde{g}\) is defined by

\[
\tilde{g}(X, Y) = g(X, JY)
\] (1.2)

and it is a Norden metric, too. Both metrics are indefinite of signature \((n, n)\).
Let $\nabla$ be the Levi-Civita connection of $g$. The tensor field $F$ of type $(0,3)$ is given by

$$F(X, Y, Z) = g((\nabla_X J)Y, Z)$$

and has the properties:

$$F(X, Y, Z) = F(X, Z, Y), \quad F(X, JY, JZ) = F(X, Y, Z).$$

The Lie 1-forms $\theta$ and $\theta^*$ associated with $F$ and the Lie vector $\Omega$ corresponding to $\theta$ are defined by:

$$\theta(x) = g^{ij}F(e_i, e_j, x), \quad \theta^* = \theta \circ J, \quad g(x, \Omega) = \theta(x),$$

where $\{e_i\} \ (i = 1, 2, \ldots, 2n)$ is an arbitrary base of the tangent space $T_pM$, $p \in M$, and $g^{ij}$ are the components of the reverse matrix of the matrix $(g_{ij})$. 
The **Nijenhuis tensor field** \( N \) for \( J \) is given by

\[
\]  

(1.6)

\[
N(X, Y) = (\nabla_X J) JY - (\nabla_Y J) JX + (\nabla_{JX} J) Y - (\nabla_{JY} J) X.
\]  

(1.7)

\[
N(X, Y, Z) = g(N(X, Y), Z)
\]  

(1.8)

It is well known [7]* that the almost complex structure is complex if and only if it is integrable, i.e. \( N = 0 \).

**The associated tensor** \( \tilde{N} \) of the Nijenhuis tensor \( N \) is defined by [2]*:

\[
\tilde{N}(X, Y) = (\nabla_X J) JY + (\nabla_Y J) JX + (\nabla_{JX} J) Y + (\nabla_{JY} J) X.
\]  

(1.9)

\[
\tilde{N}(X, Y, Z) = g(\tilde{N}(X, Y), Z)
\]  

(1.10)


A classification of the almost complex manifolds with Norden metric is introduced in [2]*, where eight classes of the considered manifolds are characterized according to the properties of $F$.

- **The class $\mathcal{W}_1$**

  \[ F(X, Y, Z) = \frac{1}{2n} \left\{ g(X, Y)\theta(Z) + g(X, Z)\theta(Y) + g(X, JY)\theta(JZ) \\
  + g(X, JZ)\theta(JY) \right\}; \]  

  \[ (1.11) \]

- **The class $\mathcal{W}_2$ of the special complex manifolds with Norden metric**

  \[ F(X, Y, JZ) + F(Y, Z, JX) + F(Z, X, JY) = 0, \quad \theta = 0; \]

  \[ (1.12) \]

- **The class $\mathcal{W}_3$ of the quasi-Kähler manifolds with Norden metric**

  \[ F(X, Y, Z) + F(Y, Z, X) + F(Z, X, Y) = 0 \iff \tilde{N} = 0; \]

  \[ (1.13) \]

• The class $\mathcal{W}_1 \oplus \mathcal{W}_2$ of the **complex manifolds with Norden metric**

\[
F(X, Y, JZ) + F(Y, Z, JX) + F(Z, X, JY) = 0 \Leftrightarrow N = 0. \quad (1.14)
\]

The special class $\mathcal{W}_0$ of the **Kähler manifolds with Norden metric** is characterized by the condition $F = 0$ and it is contained in each of the other classes.
The curvature tensor $R$ of $\nabla$ is defined by

\[
R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z, \quad R(X, Y, Z, W) = g(R(X, Y)Z, W).
\]

A tensor $L$ of type (0,4) is called **curvature-like** if it has the properties of $R$, i.e.

\[
L(X, Y, Z, W) = -L(Y, X, Z, W) = -L(X, Y, W, Z),
\]
\[
\]

Then, the Ricci tensor $\rho(L)$ and the scalar curvatures $\tau(L)$ and $\tau^*(L)$ of a curvature-like tensor $L$ are given by:

\[
\rho(L)(X, Y) = g^{ij}L(e_i, X, Y, e_j),
\]
\[
\tau(L) = g^{ij} \rho(L)(e_i, e_j), \quad \tau^*(L) = g^{ij} \rho(L)(e_i, Je_j).
\]

A curvature-like tensor $L$ is called a **Kähler tensor** if

\[
L(X, Y, JZ, JW) = -L(X, Y, Z, W).
\]
Let $S$ be a tensor of type $(0,2)$. The essential curvature-like tensors on an almost complex manifold with Norden metric are:

$$\psi_1(S)(X, Y, Z, W) = g(Y, Z)S(X, W) - g(X, Z)S(Y, W)$$
$$\quad + g(X, W)S(Y, Z) - g(Y, W)S(X, Z),$$

$$\psi_2(S)(X, Y, Z, W) = g(Y, JZ)S(X, JW) - g(X, JZ)S(Y, JW)$$
$$\quad + g(X, JW)S(Y, JZ) - g(Y, JW)S(X, JZ),$$

$$\pi_1 = \frac{1}{2}\psi_1(g), \quad \pi_2 = \frac{1}{2}\psi_2(g), \quad \pi_3 = -\psi_1(\bar{g}) = \psi_2(\bar{g}).$$

The tensors $\pi_1 - \pi_2$ and $\pi_3$ are Kählerian.
2 ALMOST COMPLEX CONNECTIONS ON ALMOST COMPLEX MANIFOLDS WITH NORDEN METRIC

**Definition 2.1** [6]* A linear connection $\nabla'$ on an almost complex manifold $(M, J)$ is called *almost complex* if $\nabla'J = 0$.

**Theorem 2.1** On an almost complex manifold with Norden metric there exists a 4-parametric family of almost complex connections $\nabla'$ with torsion tensor $T$ defined by, respectively:

$$g(\nabla'_X Y - \nabla_X Y, Z) = \frac{1}{2}F(X, JY, Z) + t_1 \{F(Y, X, Z) + F(JY, JX, Z)\}$$
$$+ t_2 \{F(Y, JX, Z) - F(JY, X, Z)\} + t_3 \{F(Z, X, Y) + F(JZ, JX, Y)\}$$
$$+ t_4 \{F(Z, JX, Y) - F(JZ, X, Y)\},$$

$$T(X, Y, Z) = t_1 \{F(Y, X, Z) - F(X, Y, Z) + F(JY, JX, Z) - F(JX, JY, Z)\}$$
$$+ (\frac{1}{2} - t_2) \{F(X, JY, Z) - F(Y, JX, Z)\} + t_2 \{F(JX, Y, Z) - F(JY, X, Z)\}$$
$$+ 2t_3 F(JZ, JX, Y) + 2t_4 F(Z, JX, Y),$$

where $t_i \in \mathbb{R}, \ i = 1, 2, 3, 4$.

Corollary 2.1 On a complex manifold with Norden metric there exists a 2-parametric family of complex connections $\nabla'$ defined by

$$
\nabla'_{X}Y = \nabla_{X}Y + \frac{1}{2}(\nabla_{X}J)JY + p\{(\nabla_{Y}J)X + (\nabla_{JY}J)JX\}
+q\{(\nabla_{Y}J)JX - (\nabla_{JY}J)X\},
$$

(2.2)

where $p = t_1 + t_3$, $q = t_2 + t_4$.

Corollary 2.2 On a quasi-Kähler manifold with Norden metric there exists a 2-parametric family of almost complex connections $\nabla'$ defined by

$$
\nabla'_{X}Y = \nabla_{X}Y + \frac{1}{2}(\nabla_{X}J)JY + s\{(\nabla_{Y}J)X + (\nabla_{JY}J)JX\}
+t\{(\nabla_{Y}J)JX - (\nabla_{JY}J)X\},
$$

(2.3)

where $s = t_1 - t_3$, $t = t_2 - t_4$. 
**Definition 2.2** [4]* A linear connection $\nabla'$ on an almost complex manifold with Norden metric $(M, J, g)$ is said to be **natural**, if

$$\nabla' J = \nabla' g = 0 \iff \nabla' g = \nabla' \tilde{g} = 0.$$ 

**Lemma 2.1** Let $(M, J, g)$ be an almost complex manifold with Norden metric and let $\nabla'$ be an arbitrary almost complex connection defined by (2.1). Then

\begin{align*}
(\nabla'_{X} g)(Y, Z) &= (t_{2} + t_{4})\tilde{N}(Y, Z, X) - (t_{1} + t_{3})\tilde{N}(Y, Z, JX), \\
(\nabla'_{X} \tilde{g})(Y, Z) &= -(t_{1} + t_{3})\tilde{N}(Y, Z, X) - (t_{2} + t_{4})\tilde{N}(Y, Z, JX). 
\end{align*}

(2.4)

**Theorem 2.2** An almost complex connection $\nabla'$ defined by (2.1) is natural on an almost complex manifold with Norden metric if and only if $t_{1} = -t_{3}$ and $t_{2} = -t_{4}$, i.e.

$$g(\nabla'_{X} Y - \nabla_{X} Y, Z) = \frac{1}{2} F(X, JY, Z) + t_{1} N(Y, Z, JX) - t_{2} N(Y, Z, X).$$

(2.5)

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The natural connection on $\mathcal{W}_1 \oplus \mathcal{W}_2$:

$$\nabla'_X Y = \nabla_X Y + \frac{1}{2}(\nabla_X J) J Y.$$  \hfill (2.6)

**Corollary 2.3** Let $(M, J, g)$ be a quasi-Kähler manifold with Norden metric. Then, the connection $\nabla'$ defined by (2.3) is natural on $M$ for all $s, t \in \mathbb{R}$. 
Definition 2.3 [4]* A natural connection $\nabla'$ with torsion tensor $T$ on an almost complex manifold with Norden metric is said to be \textit{canonical} if

$$T(X, Y, Z) + T(Y, Z, X) - T(JX, Y, JZ) - T(Y, JZ, JX) = 0. \quad (2.7)$$

Proposition 2.1 Let $(M, J, g)$ be an almost complex manifold with Norden metric. A natural connection $\nabla'$ defined by (2.5) is canonical if and only if

$$t_1 = 0, \quad t_2 = \frac{1}{8}. \quad$$

In this case (2.5) takes the form

$$2g(\nabla'_X Y - \nabla_X Y, Z) = F(X, JY, Z) - \frac{1}{4}N(Y, Z, X).$$

Remark 2.1 G. Ganchev and V. Mihova [4] have proven that on an almost complex manifold with Norden metric there exist a unique canonical connection.
Theorem 2.3 Let \((M, J, g)\) be an almost complex manifold with Norden metric and non-integrable almost complex structure. Then, on \(M\) there exists a unique almost complex connection \(\nabla'\) in the family (2.1) whose torsion tensor is totally skew symmetric (i.e. a 3-form). This connection is defined by

\[
g(\nabla'_X Y - \nabla_X Y, Z) = \frac{1}{4} \{2F(X, JY, Z) + F(Z, JX, Y) - F(JZ, X, Y)\}.
\]

Corollary 2.4 On a quasi-Kähler manifold with Norden metric there exists a unique connection \(\nabla'\) in the family of natural connections (2.3) whose torsion tensor is a 3-form. This connection is given by

\[
\nabla'_X Y = \nabla_X Y + \frac{1}{4} \{2(\nabla_X J)JY - (\nabla_Y J)JX + (\nabla_J Y J)X\}. \tag{2.8}
\]

Remark 2.2 The connection (2.8) can be considered as an analogue of the Bismut connection [1], [5]* in the geometry of the almost complex manifolds with Norden metric.

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Theorem 2.4 Let $(M, J, g)$ be a complex manifold with Norden metric. Then, on $M$ there exists a unique complex symmetric connection $\nabla'$ belonging to the family (2.2). This connection is defined by

$$\nabla'_X Y = \nabla_X Y + \frac{1}{4}\left\{ (\nabla_X J)JY + 2(\nabla_Y J)JX - (\nabla_{JX} J)Y \right\}.$$  (2.9)

Remark 2.3 The connection (2.9) is known as the Yano connection [9], [10]. In [8] we have studied this connection on a complex manifold with Norden metric belonging to the class $\mathcal{W}_1$ with closed Lie forms $\theta$ and $\theta^* = \theta \circ J$, i.e. a conformal Kähler manifold with Norden metric and we have obtained the form of its curvature tensor.


A summary of the results obtained for the family of almost complex connections $\nabla'$ defined by (2.1):

<table>
<thead>
<tr>
<th>Connection</th>
<th>$\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$</th>
<th>$\mathcal{W}_1 \oplus \mathcal{W}_2$</th>
<th>$\mathcal{W}_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>almost complex</td>
<td>$t_1, t_2, t_3, t_4 \in \mathbb{R}$</td>
<td>$p, q \in \mathbb{R}$</td>
<td>$s, t \in \mathbb{R}$</td>
</tr>
<tr>
<td>natural</td>
<td>$t_1 = -t_3$, $t_2 = -t_4$</td>
<td>$p = q = 0$</td>
<td>$s, t \in \mathbb{R}$</td>
</tr>
<tr>
<td>canonical</td>
<td>$t_1 = t_3 = 0$, $t_2 = -t_4 = \frac{1}{8}$</td>
<td>$p = q = 0$</td>
<td>$s = 0$, $t = \frac{1}{4}$</td>
</tr>
<tr>
<td>$T$ is a 3-form</td>
<td>$t_1 = t_2 = t_3 = 0$, $t_4 = \frac{1}{4}$</td>
<td>$\not\exists$</td>
<td>$s = 0$, $t = -\frac{1}{4}$</td>
</tr>
<tr>
<td>symmetric</td>
<td>$\not\exists$</td>
<td>$p = 0$, $q = \frac{1}{4}$</td>
<td>$\not\exists$</td>
</tr>
</tbody>
</table>

Table 1
3 COMPLEX CONNECTIONS ON CONFORMAL KÄHLER MANIFOLDS WITH NORDEN METRIC

\[ \nabla'_X Y = \nabla_X Y + \frac{1}{4n} \left\{ g(X, JY) \Omega - g(X, Y) J\Omega + \theta(JY) X - \theta(Y) JX \right\} \]
\[ + \frac{p}{n} \left\{ \theta(X) Y + \theta(JX) JY \right\} + \frac{q}{n} \left\{ \theta(JX) Y - \theta(X) JY \right\}. \]  

(3.1)

**Theorem 3.1** Let \((M, J, g)\) be a conformal Kähler manifold with Norden metric and \(\nabla'\) be a complex connection defined by (2.2). Then, the Kähler curvature tensor \(R'\) of \(\nabla'\) has the form

\[ R' = R - \frac{1}{4n} \left\{ \psi_1 + \psi_2 \right\} (S) - \frac{1}{8n^2} \psi_1 (P) - \frac{\theta(\Omega)}{16n^2} \left\{ 3\pi_1 + \pi_2 \right\} + \frac{\theta(J \Omega)}{16n^2} \pi_3, \]

where \(S\) and \(P\) are defined by, respectively:

\[ S(X, Y) = (\nabla_X \theta) JY + \frac{1}{4n} \left\{ \theta(X) \theta(Y) - \theta(JX) \theta(JY) \right\}, \]
\[ P(X, Y) = \theta(X) \theta(Y) + \theta(JX) \theta(JY). \]  

(3.2)
\[
\rho'(X, Y) = \rho(X, Y) - \frac{1}{4n}\{[\text{div}(J\Omega) + \frac{\theta(\Omega)}{2}]g(X, Y) - [\text{div}\Omega - \frac{\theta(J\Omega)}{2}]g(X, JY) \\
+ 2nS(X, Y) + \frac{n-1}{n}P(X, Y)\},
\]

\[
\tau' = \tau - \text{div}(J\Omega) + \frac{n-1}{4n}\theta(\Omega), \quad \tau'^* = \tau^* + \frac{n-1}{4n}\theta(J\Omega).
\]

**Theorem 3.2** Let \((M, J, g)\) be a conformal Kähler manifold with Norden metric, and \(\tau'\) and \(\tau'^*\) be the scalar curvatures of the Kähler tensor \(R'\) corresponding to the complex connection \(\nabla'\) defined by \((2.2)\). Then, the function \(\tau' + i\tau'^*\) is holomorphic on \(M\) and the Lie 1-forms \(\theta\) and \(\theta^*\) are defined in a unique way by \(\tau'\) and \(\tau'^*\) as follows:

\[
\theta = 2nd\left(\arctg\frac{\tau'}{\tau'^*}\right), \quad \theta^* = -2nd\left(\ln\sqrt{\tau'^2 + \tau'^{*2}}\right).
\]
REFERENCES

