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Lie groups as four-dimensional conformal Kähler manifolds with Norden metric

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1. Preliminaries

Let (M, J, g) be a *2n*-dimensional **almost complex manifold with Norden metric** (**almost Norden manifold**), i.e. J is an almost complex structure and g is a metric on M such that:

$$J^2X = -X, \qquad g(JX, JY) = -g(X, Y). \tag{1.1}$$

The associated metric $\tilde{g}$ of g is given by

$$\tilde{g}(X, Y) = g(X, JY) \tag{1.2}$$

and is a Norden metric, too. Both metrics are necessarily neutral, i.e. of signature *(n, n)*.

Let ∇ be the Levi-Civita connection of the metric g. Then, **the tensor field *F*** of type *(0,3)* is defined by

$$F(X, Y, Z) = g\left((\nabla_X J)Y, Z\right) \tag{1.3}$$

and has the following properties

$$F(X, Y, Z) = F(X, Z, Y) = F(X, JY, JZ). \tag{1.4}$$

The Lie forms θ and θ^*, associated with *F*, and **the Lie vector** Ω are given by

$$\theta(x) = g^{ij} F(e_i, e_j, x), \qquad \theta^* = \theta \circ J, \qquad \theta(x) = g(x, \Omega), \tag{1.5}$$

where $\{e_i\}$, $(i = 1, 2, \dots, 2n)$ is a basis of $T_p M$, $p \in M$, and x is a tangent vector.

The Nijenhuis tensor field *N* is defined by

$$N(X, Y) = [JX, JY] - [X, Y] - J[JX, Y] - J[X, JY]. \tag{1.6}$$

It is known [6]* that the almost complex structure is complex if it is integrable, i.e. if $N = 0$.

*[6] **A. Newlander**, **L. Nirenberg**, *Complex analytic coordinates in almost complex manifolds,* Ann. Math. **65** (1957), 391-404.

A classification of the almost complex manifolds with Norden metric is introduced in [2]*, where eight classes of these manifolds are characterized according to the properties of F. Another classification of these manifolds is presented in [3]*. The three basic classes W_1, W_2, W_3 and the class $W_1 \oplus W_2$ of the complex manifolds with Norden metric (Norden manifolds) are given as follows:

$$
\begin{aligned}
&W_1: \quad F(X,Y,Z)=\frac{1}{2n}\{g(X,Y)\theta(Z)+g(X,Z)\theta(Y)+g(X,JY)\theta(JZ)+g(X,JZ)\theta(JY)\};\\
&W_2: \quad F(X,Y,JZ)+F(Y,Z,JX)+F(Z,X,JY)=0, \quad \theta=0;\\
&W_3: \quad F(X,Y,Z)+F(Y,Z,X)+F(Z,X,Y)=0;\\
&W_1\oplus W_2: \quad F(X,Y,JZ)+F(Y,Z,JX)+F(Z,X,JY)=0 \;\Leftrightarrow\; N=0.
\end{aligned}
\tag{1.7}
$$

The special class W_0 of the Kähler manifolds with Norden metric, belonging to any other class, is defined by the condition $F = 0$.

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*[2] **G. Ganchev**, **A. Borisov**, *Note on the almost complex manifolds with a Norden metric*, Compt. Rend. Acad. Bulg. Sci. **39**(5) (1986), 31-34.

*[3] **G. Ganchev**, **K. Gribachev, V. Mihova**, *B-connections and their conformal invariants on conformally Kähler manifolds with B-metric*, Publ. Inst. Math. (Beograd) (N.S.) **42**(56) (1987), 107-121.

The curvature tensor *R* of ∇ is given by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \qquad (1.8)$$

The corresponding tensor of type *(0,4)* is denoted by the same letter and is defined by

$$R(X,Y,Z,W) = g\left(R(X,Y)Z,W\right). \qquad (1.9)$$

Then, **the Ricci tensor** and **the scalar curvatures** of *R* are given by

$$\rho(x,y) = g^{ij} R(e_i, x, y, e_j), \qquad \tau = g^{ij} \rho(e_i, e_j), \qquad \tau^* = g^{ij} \rho(e_i, Je_j). \qquad (1.11)$$

It is well known that **the Weyl tensor *W*** on a *2n*-dimensional pseudo-Riemannian manifold ($2n \geq 4$) is defined by

$$W = R - \frac{1}{2n-2}\left\{\psi_1(\rho) - \frac{\tau}{2n-1}\pi_1\right\}, \tag{1.12}$$

where

$$\begin{aligned}\psi_1(\rho)(X,Y,Z,W) &= g(Y,Z)\rho(X,W) - g(X,Z)\rho(Y,W)\\ &\quad + g(X,W)\rho(Y,Z) - g(Y,W)\rho(X,Z);\\ \pi_1(X,Y,Z,W) &= g(Y,Z)g(X,W) - g(X,Z)g(Y,W).\end{aligned} \tag{1.13}$$

The Weyl tensor vanishes if and only if the manifold is conformally flat.

Let $\alpha = \{x, y\}$ be a non-degenarate 2-plane spanned by vectors $x, y \in T_p M$. Then, **the sectional curvature** of α is given by

$$\nu(\alpha; p) = \frac{R(x, y, y, x)}{\pi_1(x, y, y, x)}. \tag{1.14}$$

The basic sectional curvatures in $T_p M$ are:

(i) ***holomorphic sectional curvatures*** if $J\alpha = \alpha$;

(ii) ***totally real sectional curvatures*** if $J\alpha \perp \alpha$ with respect to g.

2. Isotropic Kähler properties of complex manifolds with Norden metric

The square norm $\|\nabla J\|^2$ **of** ∇J is introduced by E. García-Río and Y. Matsushita [4]* as follows

$$\|\nabla J\|^2 = g^{ij} g^{kl} g\left((\nabla_{e_i} J)e_k, (\nabla_{e_j} J)e_l\right). \qquad (2.1)$$

Definition 2.1 [5]* An almost complex manifold with Norden metric, satisfying the condition $\|\nabla J\|^2 = 0$, is called ***an isotropic Kähler manifold with Norden metric.***

Theorem 2.1 On a complex manifold with Norden metric the square norm of ∇J satisfies

$$\tau + \tau^{**} + \theta(\Omega) - 2\,\mathrm{div}(J\Omega) = \frac{1}{2}\|\nabla J\|^2, \qquad (2.2)$$

where $\tau^{**} = g^{il} g^{ik} R(e_i, e_j, Je_k, Je_l)$ and $\mathrm{div}(J\Omega) = g^{ij}(\nabla_{e_i}\theta^*)e_j$.

*[4] **E. García-Río**, **Y. Matsushita**, *Isotropic Kähler structures on Engel 4-manifolds*, J. Geom. Phys. **33** (2000), 288-294.

*[5] **K. Gribachev**, **M. Manev**, **D. Mekerov**, *A Lie group as a 4-dimensional quasi-Kähler manifold with Norden metric,* JP J. Geom. Topol. **6**(1) (2006), 55-68.

It is proved in [7]* that on a W_1-manifold $\|\nabla J\|^2 = \frac{2}{n}\theta(\Omega)$.

Corollary 2.2 On a W_1-manifold with Norden metric it is valid

$$\tau + \tau^{**} - 2\,\mathrm{div}(J\Omega) = -\frac{n-1}{2}\|\nabla J\|^2 . \qquad (2.3)$$

Corollary 2.3 A W_1-manifold is isotropic Kählerian if and only if the condition

$$\tau + \tau^{**} = 2\,\mathrm{div}(J\Omega) \qquad (2.4)$$

holds.

Corollary 2.4 On a W_2-manifold with Norden metric we have

$$2(\tau + \tau^{**}) = \|\nabla J\|^2 . \qquad (2.5)$$

Corollary 2.5 A W_2-manifold is isotropic Kählerian if its curvature tensor is Kählerian, i.e. if $R(X,Y,JZ,JW) = -R(X,Y,Z,W)$.

*[7] **M. Teofilova**, *Curvature properties of conformal Kähler manifolds with Norden metric,* Math. Educ. Math., Proc. of 35th Spring Conference of UBM, Borovec (2006), 214-219.

3. A Lie group as a 4-dimensional conformal Kähler manifold with Norden metric

Let $\mathfrak{g}$ be a real 4-dimensional Lie algebra corresponding to a real connected Lie group G. If $\{X_1, X_2, X_3, X_4\}$ is a global basis of left invariant vector fields on G, and $[X_i, X_j] = C_{ij}^k X_k$, then the Jacobi identity is valid

$$C_{ij}^k C_{ks}^l + C_{js}^k C_{ki}^{\ l} + C_{si}^k C_{kj}^l = 0. \tag{3.1}$$

We can define **an almost complex structure** on G by the conditions

$$JX_1 = X_3, \quad JX_2 = X_4, \quad JX_3 = -X_1, \quad JX_4 = -X_2. \tag{3.2}$$

Further, let us consider **the left invariant metric** given by

$$\begin{aligned} &g(X_1, X_1) = g(X_2, X_2) = -g(X_3, X_3) = -g(X_4, X_4) = 1, \\ &g(X_i, X_j) = 0, \quad i \neq j. \end{aligned} \tag{3.3}$$

In this way the induced 4-dimensional manifold (G, J, g) is **an almost complex manifold with Norden metric**.

Definition 3.1 [1]* An almost complex structure J on a Lie group G is said to be ***abelian*** if

$$[JX, JY] = [X, Y] \qquad \text{for all} \quad X, Y \in \mathfrak{g}. \tag{3.4}$$

From (3.4) it follows that the Nijenhuis tensor vanishes on $\mathfrak{g}$, i.e. $N = 0$ and thus (G, J, g) is **a complex manifold with Norden metric**.

It has been proved in [1]* that any Lie algebra admitting an abelian complex structure is solvable.

*[1] **I. Dotti**, *Hypercomplex nilpotent Lie groups*, Contemporary Math. **288** (2001), 310-314.

Lemma 3.1 Let (G, J, g) be a 4-dimensional W_1-manifold with Norden metric, admitting an abelian complex structure. Then, the Lie algebra $\mathfrak{g}$ of G is determined by the conditions:

$$\begin{array}{llll} C_{13}^1 = C_{14}^2 - C_{12}^4, & C_{13}^2 = C_{12}^3 - C_{14}^1, & C_{13}^3 = C_{12}^2 + C_{14}^4, & C_{13}^4 = -C_{12}^1 - C_{14}^3, \\ C_{24}^1 = -C_{12}^4 - C_{14}^2, & C_{24}^2 = C_{12}^3 + C_{14}^1, & C_{24}^3 = C_{12}^2 - C_{14}^4, & C_{24}^4 = C_{14}^3 - C_{12}^1, \end{array} \tag{3.6}$$

where $C_{ij}^k \in \mathbb{R}$ (i, j, k=1,2,3,4) must satisfy the Jacobi identity.

The Lie algebra $\mathfrak{g}$ of G

$$\begin{aligned} &[X_1, X_2] = [X_3, X_4] = 0, \\ &[X_1, X_4] = [X_2, X_3] = \lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3 + \lambda_4 X_4, \\ &[X_1, X_3] = -[X_2, X_4] = \lambda_2 X_1 - \lambda_1 X_2 + \lambda_4 X_3 - \lambda_3 X_4. \end{aligned} \tag{3.7}$$

Thus, $(G, J, g) \in W_1$.

Geometric characteristics of (G, J, g)

The non-zero components of the Levi-Civita connection

$$\nabla_{X_1} X_1 = \nabla_{X_2} X_2 = \lambda_2 X_3 + \lambda_1 X_4, \qquad -\nabla_{X_3} X_3 = -\nabla_{X_4} X_4 = \lambda_4 X_1 + \lambda_3 X_2,$$
$$\nabla_{X_1} X_3 = \nabla_{X_4} X_2 = \lambda_2 X_1 - \lambda_3 X_4, \qquad \nabla_{X_1} X_4 = -\nabla_{X_3} X_2 = \lambda_1 X_1 + \lambda_3 X_3, \tag{3.8}$$
$$\nabla_{X_2} X_3 = -\nabla_{X_4} X_1 = \lambda_2 X_2 + \lambda_4 X_4, \qquad \nabla_{X_2} X_4 = \nabla_{X_3} X_1 = \lambda_1 X_2 - \lambda_4 X_3.$$

The essential non-zero components of the tensor *F*

Let us denote $F_{ijk} = F(X_i, X_j, X_k)$.

$$\frac{1}{2} F_{222} = F_{112} = F_{314} = \lambda_1, \qquad \frac{1}{2} F_{111} = F_{212} = -F_{414} = \lambda_2,$$
$$\frac{1}{2} F_{422} = -F_{114} = F_{312} = -\lambda_3, \qquad \frac{1}{2} F_{311} = F_{214} = F_{412} = -\lambda_4. \tag{3.9}$$

The components of the Lie forms θ and θ^*

Let us denote $\theta_i = \theta(X_i)$ and $\theta_i^* = \theta^*(X_i)$.

$$\begin{aligned} &\theta_1 = 4\lambda_2, \qquad \theta_2 = 4\lambda_1, \qquad \theta_3 = 4\lambda_4, \qquad \theta_4 = 4\lambda_3; \\ &\theta_1^* = 4\lambda_4, \qquad \theta_2^* = 4\lambda_3, \qquad \theta_3^* = -4\lambda_2, \qquad \theta_4^* = -4\lambda_1. \end{aligned} \tag{3.10}$$

A *1*-form ω is said to be ***closed*** if $d\omega = 0$. For the Levi-Civita connection the following equivalent condition holds

$$(\nabla_X \omega)Y = (\nabla_Y \omega)X. \tag{3.11}$$

A W_1-manifold with both closed Lie forms θ and θ^* is called ***a conformal Kähler manifold with Norden metric***. These manifolds are conformally equivalent to Kähler manifolds with Norden metric.

Proposition 3.2 The Lie form θ^* is closed on (G, J, g).

Proposition 3.3 The manifold (G, J, g) is conformal Kählerian if and only if the Lie form θ is closed, i.e. if and only if one of the conditions holds:

(i) $\lambda_1 = \lambda_4, \quad \lambda_2 = -\lambda_3$;

(ii) $\lambda_1 = -\lambda_4, \quad \lambda_2 = \lambda_3$.

The square norm of ∇J

$$\| \nabla J \|^2 = 16\left(\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2\right). \tag{3.12}$$

Proposition 3.4 The manifold (G, J, g) is isotropic Kählerian if and only if the condition

$$\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2 = 0 \tag{3.13}$$

holds.

Proposition 3.5 If (G, J, g) is a conformal Kähler manifold with Norden metric, then it is isotropic Kählerian.

Curvature properties of (G, J, g)

The non-zero components of the curvature tensor *R*

Let us denote $R_{ijkl} = R(X_i, X_j, X_k, X_l)$.

$$
\begin{aligned}
&R_{1221} = \lambda_1^2 + \lambda_2^2, && R_{1341} = R_{2342} = -\lambda_1\lambda_2,\\
&R_{1331} = \lambda_4^2 - \lambda_2^2, && R_{2132} = -R_{4134} = -\lambda_1\lambda_3,\\
&R_{1441} = \lambda_4^2 - \lambda_1^2, && R_{1231} = -R_{4234} = \lambda_1\lambda_4,\\
&R_{2332} = \lambda_3^2 - \lambda_2^2, && R_{2142} = -R_{3143} = \lambda_2\lambda_3,\\
&R_{2442} = \lambda_3^2 - \lambda_1^2, && R_{1241} = -R_{3243} = -\lambda_2\lambda_4,\\
&R_{3443} = -\lambda_3^2 - \lambda_4^2, && R_{3123} = R_{4124} = \lambda_3\lambda_4.
\end{aligned}
\qquad (3.14)
$$

The components of the Ricci tensor

Let us denote $\rho_{ij} = \rho(X_i, X_j)$.

$$\begin{aligned}
&\rho_{11} = 2\left(\lambda_1^2 + \lambda_2^2 - \lambda_4^2\right), && \rho_{12} = -2\lambda_3\lambda_4, && \rho_{23} = 2\lambda_1\lambda_4,\\
&\rho_{22} = 2\left(\lambda_1^2 + \lambda_2^2 - \lambda_3^2\right), && \rho_{13} = -2\lambda_1\lambda_3, && \rho_{24} = -2\lambda_2\lambda_4,\\
&\rho_{33} = 2\left(\lambda_4^2 + \lambda_3^2 - \lambda_2^2\right), && \rho_{14} = 2\lambda_2\lambda_3, && \rho_{34} = -2\lambda_1\lambda_2.\\
&\rho_{44} = 2\left(\lambda_4^2 + \lambda_3^2 - \lambda_1^2\right), && &&
\end{aligned} \tag{3.15}$$

Proposition 3.6 The manifold (G, J, g) is Ricci-symmetric, i.e. $\nabla\rho = 0$.

The values of the scalar curvatures

$$\tau = 6\left(\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2\right), \qquad \tau^* = -4\left(\lambda_1\lambda_3 + \lambda_2\lambda_4\right). \tag{3.16}$$

Proposition 3.7 For the manifold (G, J, g) it is valid:

(i) (G, J, g) is isotropic Kählerian if and only if $\tau = 0$;

(ii) (G, J, g) is conformal Kählerian if and only if $\tau = \tau^* = 0$.

Proposition 3.8 The Weyl tensor of (G, J, g) vanishes, i.e. $W = 0$.

Proposition 3.9 The curvature tensor R of (G, J, g) has the form

$$R = \frac{1}{2}\left\{\psi_1(\rho) - \frac{\tau}{3}\pi_1\right\}. \tag{3.17}$$

Proposition 3.10 If (G, J, g) a conformal Kähler manifold, then its curvature tensor has the form

$$R = \frac{1}{2}\psi_1(\rho). \tag{3.18}$$

Proposition 3.11 The manifold (G, J, g) is locally symmetric, i.e. $\nabla R = 0$.

The values of the sectional curvatures

Let us consider the characteristic 2-planes α_{ij} spanned by the basic vectors $\{X_i, X_j\}$ at an arbitrary point of the manifold:

(i) totally real 2-planes: $\alpha_{12}, \alpha_{14}, \alpha_{23}, \alpha_{34}$;

(ii) holomorhic 2-planes: α_{13}, α_{24}.

$$\begin{array}{lll} \nu(\alpha_{12}) = \lambda_1^2 + \lambda_2^2, & \nu(\alpha_{23}) = \lambda_2^2 - \lambda_3^2, & \nu(\alpha_{13}) = \lambda_2^2 - \lambda_4^2, \\ \nu(\alpha_{14}) = \lambda_1^2 - \lambda_4^2, & \nu(\alpha_{34}) = -\lambda_3^2 - \lambda_4^2, & \nu(\alpha_{24}) = \lambda_1^2 - \lambda_3^2. \end{array} \tag{3.19}$$

Proposition 3.12 If the manifold (G, J, g) is of vanishing holomorphic sectional curvatures, then it is isotropic Kählerian.

Proposition 3.13 The following conditions are equivalent for the manifold (G, J, g):

(i) (G, J, g) is isotropic Kählerian;

(ii) the condition $\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2 = 0$ holds;

(iii) the scalar curvature τ vanishes;

(iv) the curvature tensor has the form $R = \frac{1}{2}\psi_1(\rho)$.

References

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2. **G. Ganchev**, **A. Borisov**, *Note on the almost complex manifolds with a Norden metric*, Compt. Rend. Acad. Bulg. Sci. **39**(5) (1986), 31-34.
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